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Interaction of strongly nonlinear waves on the free surface of non-conducting fluid under the action of horizontal electric field

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The purpose of the work is theoretical study of the effect of the horizontal electric field on strongly nonlinear dynamics of the free surface of a non-conductive fluid.

To achieve the above purpose is necessary to solve the following problems:

1. To choose an effective method for the numerical investigation of the free surface dynamics, which allows to describe the formation of regions with high steepness (e.g. cusps or fingers);
2. To investigate the interaction between strongly nonlinear waves on the surface of the non-conducting liquid with a high dielectric constant in a horizontal electric field.

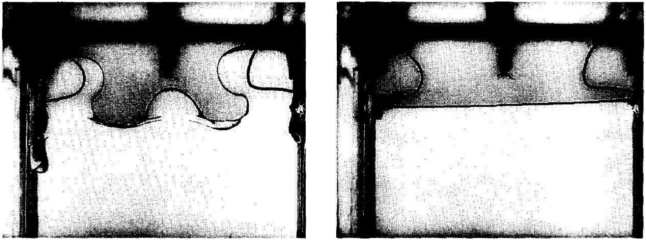


Figure 1: Stabilization of the Rayleigh-Taylor instability by the external electric field; see Zahn M., Haus H.A. Journal of Electrostatics, 1995. **34**. P. 109-162

The initial equations of motion

The equations of motion of an ideal fluid with a free surface in an external tangential electric field have the following form

$$\Delta\phi = 0, \qquad \Delta\varphi = 0,$$

$$\phi_t + (\nabla\phi)^2/2 = (P_E - P_0)/\rho, \qquad y = \eta(x,t)$$

$$\eta_t = \phi_y - \eta_x\phi_x, \qquad y = \eta(x,t)$$

$$\varphi_y - \eta_x\varphi_x = 0, \qquad y = \eta(x,t)$$

$$\phi \to 0, \qquad \varphi \to -Ex, \qquad y \to -\infty$$

where $P_E = \varepsilon\varepsilon_0(\nabla\varphi)^2/2$, is the electrostatic pressure, $P_0 = \varepsilon\varepsilon_0 E^2/2$ is the energy density of external electric field in the liquid. For convenient further consideration, we pass to the dimensionless variables as

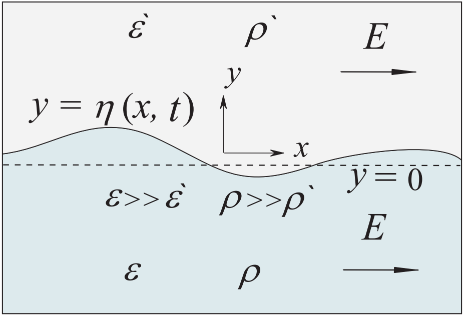


Figure 2: The geometry of the problem is shown schematically

$$\phi \to \lambda E\varepsilon_0^{1/2}\varepsilon^{1/2}\rho^{-1/2}\phi, \quad t \to \lambda E^{-1}\varepsilon_0^{1/2}\varepsilon^{-1/2}\rho^{1/2}t, \quad \varphi \to \lambda E\varphi, \quad \mathbf{r} \to \lambda\mathbf{r}.$$

It is known that the system of the initial equations has a solution in the form of a traveling wave without dispersion in the direction of, or against the direction of the external electric field.

Interaction of counterpropagating weakly nonlinear waves is described by the expression

$$\eta(t,x) = f + g + \partial_x \hat{H}(\hat{H}f \cdot \hat{H}g) + O(\alpha^3),$$

where $f = f(x-t)$, $g = g(x+t)$ are localized functions, $\hat{H}$ is the Hilbert transform

$$\hat{H}f = \frac{1}{\pi} P.V. \int_{-\infty}^{+\infty} \frac{f(u')}{u-u'} du'.$$

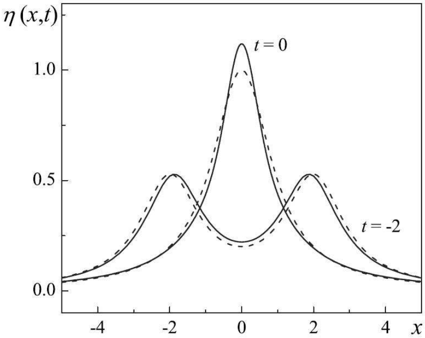


Figure 3: Evolution of counterpropagating waves; see N.M. Zubarev, O.V. Zubareva, Phys. Rev. E **82**, 046301 (2010)

The counterpropagating solitary weakly nonlinear waves preserve the initial form after collision. The question arises as to whether the same tendency is observed for strongly nonlinear waves. The answer on this question requires solving the fully nonlinear system of the equations of motion.

Conformal transform

We make the conformal transform of the region occupied by the fluid to the parametric half-plane as shown at Fig. 4. In the new variables, the Laplace equations for the electric field potential and velocity potential can be solved analytically. As a result, the initial problem of motion of the liquid can be reduced to the problem of motion of its free surface, which has lower dimension of 1+1.

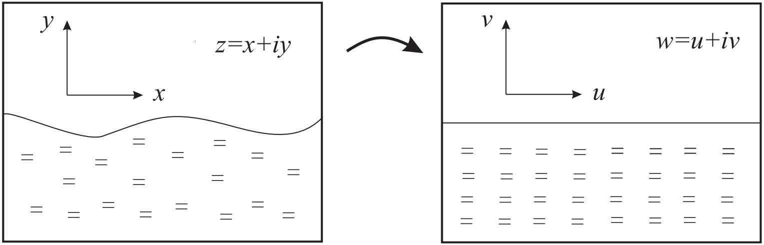


Figure 4: The conformal transform: $(x, y) \to (u, v)$

The surface of the liquid in the new variables is specified by the parametric expressions

$$y = Y(u, t), \qquad x = X(u, t) = u - \hat{H}Y.$$

The equations in the conformal variables

The equations of the boundary motion are represented in Hamiltonian form, and the functions η and $\psi = \phi|_{y=\eta}$ are canonically conjugate variables

$$\psi_t = -\frac{\delta H}{\delta \eta}, \qquad \eta_t = \frac{\delta H}{\delta \psi}.$$

The Hamiltonian H and horizontal momentum P of the system are expressed in terms of parameterized quantities

$$H = -\frac{1}{2}\int_{-\infty}^{+\infty} (\Psi \hat{H} \Psi_u + Y \hat{H} Y_u)\, du, \qquad P = \int_{-\infty}^{+\infty} \Psi Y_u\, du,$$

where the function $\Psi(u, t)$ defines the value of the velocity potential at the boundary $v = 0$.

In order to obtain the equations of motion resolved with respect to time derivatives, let us introduce the complex functions

$$Z = X + iY = u + 2i\hat{P}Y, \qquad \Phi = 2\hat{P}\Psi,$$

which are analytical in lower half-plan of the conformal variable u (here $\hat{P} = (1 + i\hat{H})/2$ is projector). As a result, the equations of motion take the form

$$Z_t = iUZ_u,$$

$$\Phi_t = iU\Phi_u - B + \hat{P}\,(1/J - 1)\,,$$

where $J = X_u^2 + Y_u^2$ is the Jacobian, and $U = -2\hat{P}\left(\hat{H}\Psi_u/J\right),\ B = \hat{P}\left(\Phi_u\overline{\Phi}_u/J\right).$

The equations in Dyachenko variables

In order to eliminate the Jacobian from the denominators let us pass to the new (Dyachenko) variables R, V

$$R = \frac{1}{Z_u}, \qquad V = i\frac{\Phi_u}{Z_u},$$

here Z, Φ are the complex function determining the surface profile and the velocity potential, respectively. As a result, the equations of motion take the symmetric form

$$R_t = i(UR_u - U_u R), \qquad V_t = i(UV_u - D_u R),$$

$$U = \hat{P}(V\overline{R} + \overline{V}R), \qquad D = \hat{P}(V\overline{V} - R\overline{R}),$$

The numerical scheme is based on the key properties of the operators $\hat{P}$ and ∂_u

$$\hat{P}\left(\sum_{k=-\infty}^{+\infty} a^k e^{-iku}\right) = \frac{a_0}{2} + \sum_{k=1}^{+\infty} a^k e^{-iku}, \qquad \partial_u\left(\sum_{k=-\infty}^{+\infty} a^k e^{-iku}\right) = -\sum_{k=-\infty}^{+\infty} ik \cdot a^k e^{-iku}.$$

a^k are the Fourier coefficients of a function $a(u)$.

A comparison with the exact solution

Results of the comparison of numerical and analytical solutions, describing the motion of a traveling wave are presented. The initial conditions were $R(u) = 1 + 0.5\exp(-iu), V(u) = -0.5i\exp(-iu)$.

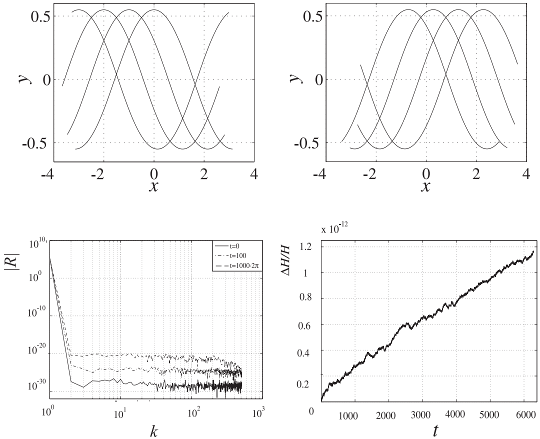


Figure 5: Surface profile, spectrum and calculations error. The number of negative harmonics is $N = 512$, the time step $\tau = 10^{-3}$.

Let us present the results of the comparison of numerical and analytical solutions, describing the interaction between nonlinear waves with relatively small amplitude

$$f = a\exp(-2(x-t)^2), \qquad g = a\exp(-2(x+t)^2), \qquad a \ll 1$$

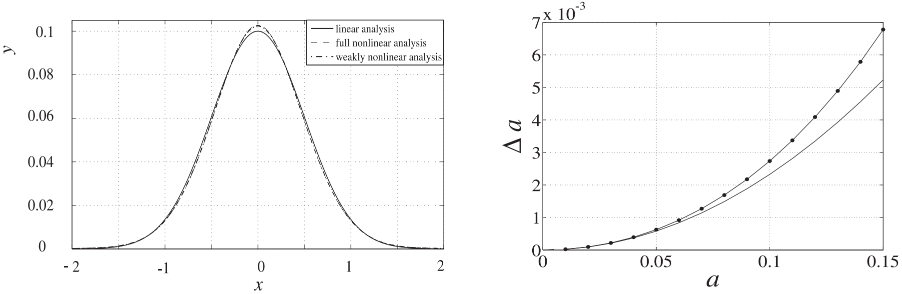
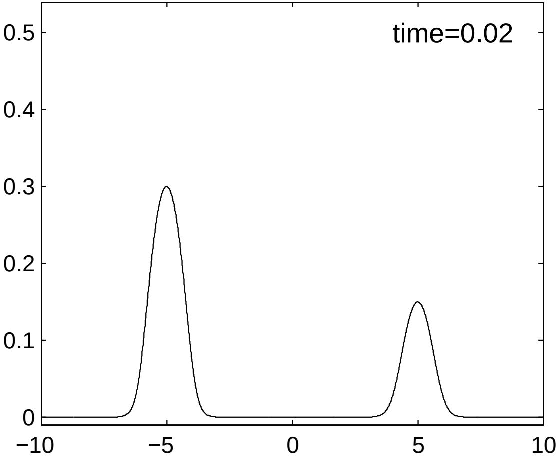


Figure 6: The surface profile of the liquid at the moment of collision of symmetric waves (left figure). The dependence of the amplitude jump Δa on the wave linear amplitude a, the solid line corresponds to the analytical solution and the dots show the results of numerical simulations (right figure), $N = 1024, \tau = 10^{-3}$

The interaction of strongly nonlinear waves

The animation are showing the result of simulation the interaction between strongly nonlinear waves. We can see that the waves do not recover their initial form after collision, in contrast to the weakly nonlinear waves. Initially symmetric waves are tilted in the direction opposite to their motion.



Elastic interaction

The numerical data have identified an amazing feature: the energy and momentum of each wave is preserved during the interaction, i.e. the interaction is elastic. To proof it, let us introduce the auxiliary functions

$$F^{\pm}(u,t) = (Y \pm \hat{H}\Psi)/2.$$

For the particular solution corresponding to a wave moving to the right $F^{+} = Y^{+}$ and $F^{-} = 0$. For a wave moving to the left $F^{+} = 0 \;\; F^{-} = Y^{-}$.

In terms of $F^{\pm}$ the energy of the system H and horizontal momentum P are rewritten as

$$H = H^{+} + H^{-}, \qquad P = P^{+} + P^{-}, \qquad H^{\pm} = \mp P^{\pm} = -\int\limits_{-\infty}^{+\infty} F^{\pm}\hat{H}F_{u}^{\pm}\,du.$$

Each of the quantities $H^{\pm}$ and $P^{\pm}$ is an integral of motion. Indeed, it is easy to verify that

$$H^{\pm} = (H \pm P)/2, \qquad P^{\pm} = \mp(H \mp P)/2,$$

i.e. the energies and momenta of individual waves are represented as combinations of invariants of H and P.

Deformation of the strongly nonlinear waves

It can be seen that waves propagate without distortions before and after the time of collision at $x = 0$. Intense interaction between waves occurs at the time of collision. As a result, the shape of the waves changes: they are tilted in the direction opposite to their motion.

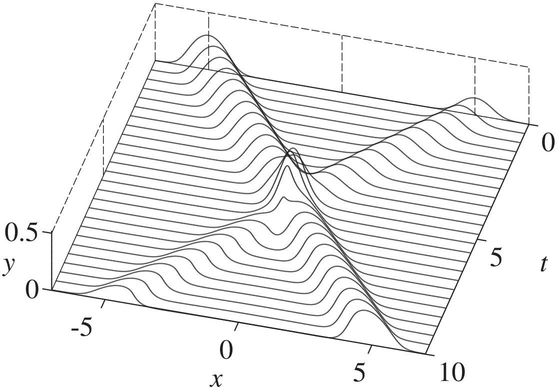


Figure 7: The result of numerical simulation of a single collision of the counterpropagating waves.

Deformation of the strongly nonlinear waves

Let us introduce the parameter of "asymmetry" of a function $S = \min(\partial f/\partial u) + \max(\partial f/\partial u)$

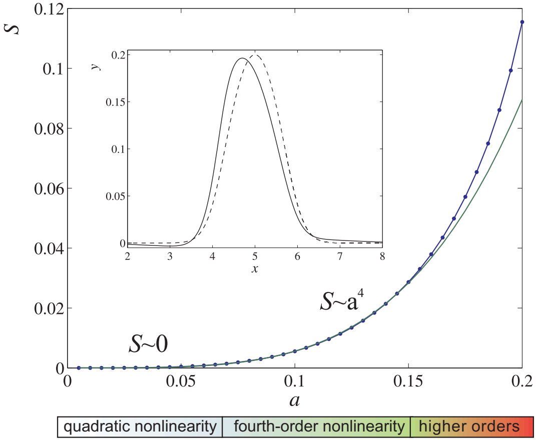
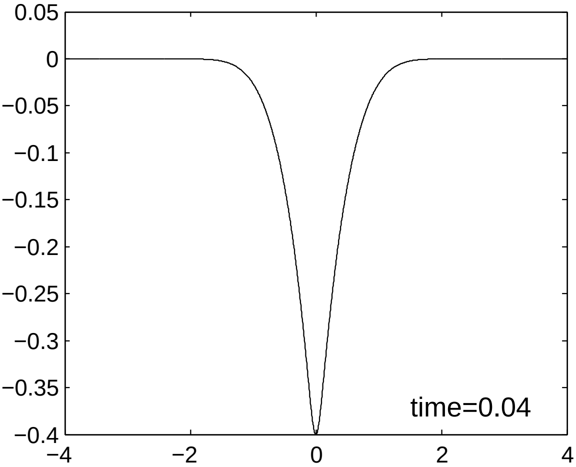


Figure 8: Parameter S versus the amplitude a. Points on the blue line are the results of a series of numerical experiments. The green is the dependence $S = 60a^4$. The inset shows the wave profile with $a = 0.2$ (dashed line) before and (solid line) after collision.

Multiple interactions of the waves

Periodic collisions of the waves lead to the formation of regions with high surface curvature $y_c = y_{xx}/(1+y_x)^{3/2}$.



Singularities formation

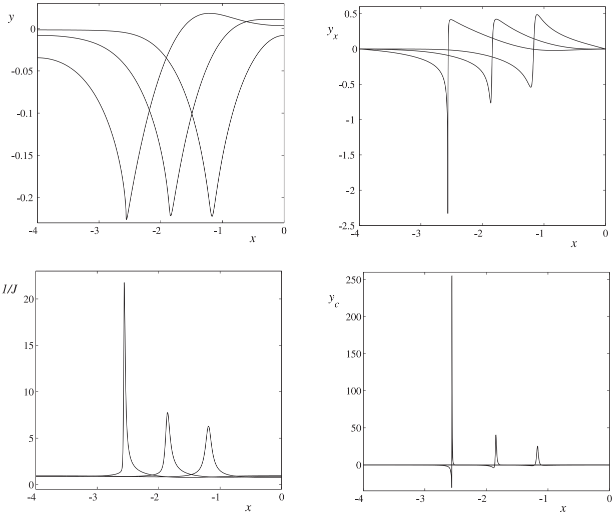


Figure 9: The figure shows the profile of the solitary wave moving leftward, the energy density of electric field, the steepness and curvature of the surface at the successive times $t_1 = 1.20,\ t_2 = 10.10,\ t_3 = 19.05$.

Conclusion

It has been demonstrated that the energy and horizontal momentum of each of the oppositely propagating interacting solitary waves are conserved; i.e., their interaction is elastic. The numerical solution of the equations of motion in the conformal variables has shown that the shape of waves changes. In particular, the trailing edge of waves with a positive amplitude becomes steeper. It has been demonstrated that the deformation of waves is determined by fourth order nonlinearity. This process has an accumulative character: tendency to the formation of singularities, which are points with a large curvature of the surface, is observed at multiple collisions of waves.

References

- N. M. Zubarev, E.A. Kochurin, JETP Lett. **99**, 729 (2014).
- V.E. Zakharov, R.V. Shamin, A.V. Yudin, JETP Lett. **99**, 514 (2014).
- N. M. Zubarev and O. V. Zubareva, Tech. Phys. Lett. **32**, 886 (2006).
- N. M. Zubarev, JETP Lett. **89**, 271 (2009).
- A.I. Dyachenko, E.A. Kuznetsov, M.D. Spector, V.E. Zakharov, Phys. Lett. A **221**, 73 (1996).
- A. I. Dyachenko, Dokl. Math. **63**, 115 (2001).
- V.E. Zakharov, A.I. Dyachenko, O.A. Vasilyev, Eur. J. Mech. B - Fluids **21**, 283 (2002).
- B. Tao, D.L. Guo, Comput. Math. Appl. **67**, 627 (2014).
- N.M. Zubarev, O.V. Zubareva, Phys. Rev. E **82**, 046301 (2010).
- P. Goldreich, S. Sridhar, Astrophys. J. **438**, 763 (1995).

Thank you for your attention!

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Acknowledgements

Taking into account the gravity and capillary the system of the governing equations takes slightly more complicated form

$$Q_t = i(UQ_u - \frac{1}{2}U_u Q),$$

$$V_t = i(UV_u - Q^2 B_u) + g(Q^2 - 1) - 2\sigma Q^2 \hat{P}(Q_u \overline{Q} - \overline{Q}_u Q)_u,$$

where g, σ — dimensionless acceleration of gravity and surface tension, respectively, and the functions Q, V, B defined as

$$Q = \sqrt{R}, \qquad U = \hat{P}(V\overline{Q}^2 + \overline{V}Q^2), \qquad B = \hat{P}(V\overline{V} - (Q\overline{Q})^2).$$