Analytic theory of wind-driven sea

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Results obtained in collaboration with S.I. Badulin, V.V. Geogjaev, A.N. Pushkarev are used in this talk

In this talk

1. New form of coupling coefficient.
2. Asymptotics for long–short wave interactions
3. Diffusion equation for long–short wave interaction
4. Powerlike solutions and the Kolmogorov constants
5. Asymptotics for nonlinear damping
6. Phillips-like dissipation term
7. Self-similar solutions. Magic relations.
8. Quadruplet form of the Hasselmann equation
9. New numerical algorithm for solving the Hasselmann equation
10. Comparison with Resio–Tracy and DIA methods

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What Hasselmann equation is written for?
$Q_{\mathbf{k}\omega} =< |\eta_{\mathbf{k}\omega}|^2 >$ — space-time spectrum of the surface.
In the linear approximation

$$Q_{\mathbf{k}\omega} = \frac{\omega_k}{2} \left(N_{\mathbf{k}} \delta(\omega - \omega_k) + N_{-\mathbf{k}} \delta(\omega + \omega_k) \right)$$

In reality

$$N_{\mathbf{k}} = N_{0\mathbf{k}} + \mu^2 N_{1\mathbf{k}} + \dots$$

(μ is average steepness)
$N_{1\mathbf{k}}$ includes "bound harmonics"

$$\delta(\omega - \omega_k) \to \frac{1}{\pi} \frac{\Gamma_{\mathbf{k}}}{(\omega - \tilde{\omega}_k)^2 + \Gamma_{\mathbf{k}}^2}$$

$$\tilde{\omega}_k = \omega_k + \mu^2 \omega_{1k} + \dots$$

(μ is average steepness)

The Hasselmann equation reads

$$\frac{dN}{dt} = \frac{\partial N}{\partial t} + \frac{\partial \omega}{\partial k}\,\frac{\partial N}{\partial t} = S_{\mathrm{nl}}$$

$$S_{\mathrm{nl}} = F_{\mathbf{k}} - \Gamma_{\mathbf{k}} N_{\mathbf{k}}$$

$$F_{\mathbf{k}} = \pi g^2 \int\limits_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3} (T_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3})^2\, N_{\mathbf{k}_1} N_{\mathbf{k}_2} N_{\mathbf{k}_3} \;\times$$

$$\times\, \delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)\, \delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)\, d\mathbf{k}_1\, d\mathbf{k}_2\, d\mathbf{k}_3$$

$$\Gamma_{\mathbf{k}} = \pi g^2 \int\limits_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3} (T_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3})^2\, (N_{\mathbf{k}_1} N_{\mathbf{k}_2} + N_{\mathbf{k}_1} N_{\mathbf{k}_3} - N_{\mathbf{k}_2} N_{\mathbf{k}_3}) \;\times$$

$$\times\, \delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)\, \delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)\, d\mathbf{k}_1\, d\mathbf{k}_2\, d\mathbf{k}_3$$

New form of coupling coefficient

$$
\begin{aligned}
\tilde{T}_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\mathbf{k}_4} \quad = \quad & -\frac{1}{4}\frac{1}{(k_1k_2k_3k_4)^{1/4}}\Bigg\{ \\
& +\frac{1}{2}\left(k_{1+2}^2-(\omega_1+\omega_2)^4\right)\left[(\mathbf{k}_1\mathbf{k}_2-k_1k_2)+(\mathbf{k}_3\mathbf{k}_4-k_3k_4)\right] \\
& -\frac{1}{2}\left(k_{1-3}^2-(\omega_1-\omega_3)^4\right)\left[(\mathbf{k}_1\mathbf{k}_3+k_1k_3)+(\mathbf{k}_2\mathbf{k}_4+k_2k_4)\right] \\
& -\frac{1}{2}\left(k_{1-4}^2-(\omega_1-\omega_4)^4\right)\left[(\mathbf{k}_1\mathbf{k}_4+k_1k_4)+(\mathbf{k}_2\mathbf{k}_3+k_2k_3)\right] \\
& +\left(\frac{4(\omega_1+\omega_2)^2}{k_{1+2}-(\omega_1+\omega_2)^2}-1\right)(\mathbf{k}_1\mathbf{k}_2-k_1k_2)(\mathbf{k}_3\mathbf{k}_4-k_3k_4) \\
& +\left(\frac{4(\omega_1-\omega_3)^2}{k_{1-3}-(\omega_1-\omega_3)^2}-1\right)(\mathbf{k}_1\mathbf{k}_3+k_1k_3)(\mathbf{k}_2\mathbf{k}_4+k_2k_4) \\
& +\left(\frac{4(\omega_1-\omega_4)^2}{k_{1-4}-(\omega_1-\omega_4)^2}-1\right)(\mathbf{k}_1\mathbf{k}_4+k_1k_4)(\mathbf{k}_2\mathbf{k}_3+k_2k_3)\Bigg\}
\end{aligned}
$$

The above form should be symmetrized:

$$T_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4} = \frac{1}{2}\left(\tilde{T}_{1234} + \tilde{T}_{2143}\right)$$

The diagonal part

$$\begin{aligned} T_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_1\mathbf{k}_2} \quad = \quad & -\frac{1}{2}\frac{1}{(k_1 k_2)^{1/2}}\Bigg\{3k_1^2 k_2^2 + (\mathbf{k}_1\mathbf{k}_2)^2 \\ & - 4\omega_1\omega_2\,\mathbf{k}_1\mathbf{k}_2\,(k_1 + k_2)\;\frac{1}{g^2} \\ & + 2(\omega_1+\omega_2)^2\frac{(\mathbf{k}_1\mathbf{k}_2 - k_1 k_2)^2}{gk_{1+2} - (\omega_1+\omega_2)^2} \\ & + 2(\omega_1-\omega_2)^2\frac{(\mathbf{k}_1\mathbf{k}_2 + k_1 k_2)^2}{gk_{1-2} - (\omega_1-\omega_2)^2}\Bigg\} \end{aligned}$$

Asymptotics

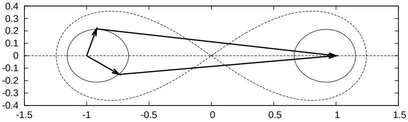
$$\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3 + \mathbf{k}_4$$

$$\omega_1 + \omega_2 = \omega_3 + \omega_4$$

Suppose $k_1 \ll k_2$, $k_3 \ll k_4$. In this case

$$\mathbf{k}_2 \approx \mathbf{k}_4$$

$$|k_1| \approx |k_3|$$



After some lengthy algebra we get the asymptotic value of T:

$$T_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4)} = -\frac{1}{2}k_1^2 k_2 \, T_{\theta_1\theta_2}$$

$$T_{\theta_1\theta_2} = (\cos\theta_1 + \cos\theta_3)\cos(\theta_1 - \theta_3)$$

Diffusion equation

Let $N = N_0 + N_1$
N_0 — low frequency

$$\frac{\partial N_1}{\partial t} = \frac{\partial}{\partial k_i} D_{ij} k^2 \frac{\partial}{\partial k_i} N$$

$$D_{ij} = 2\pi g^{3/2} \int_0^\infty dq\, q^{17/2} \int_0^{2\pi} d\theta_1 \int_0^{2\pi} d\theta_3 |T(\theta_1, \theta_3)|^2\, p_i p_j N(\theta, q) N(\theta_3, q)$$

$$p_1 = \cos\theta_1 - \cos\theta_3, \quad p_2 = \sin\theta_1 - \sin\theta_3$$

In the isotropic case

$$\frac{\partial N_1}{\partial t} = \frac{D}{k} \frac{\partial}{\partial k} k^3 \frac{\partial N_1}{\partial k}$$

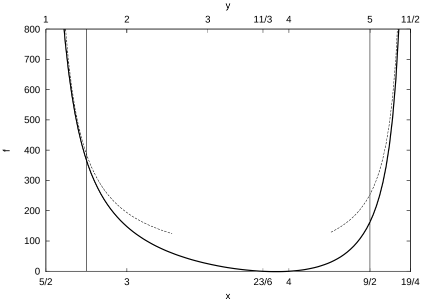
Powerlike spectra

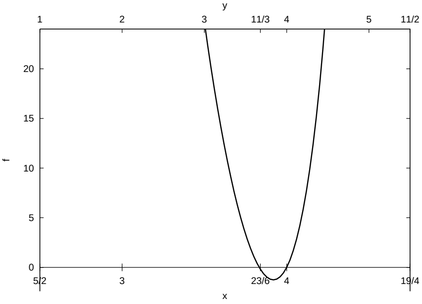
For $N = k^{-x}$ we define the function F such that

$$S_{\mathrm{nl}} = g^{\frac{3}{2}} k^{-3x+\frac{19}{2}} F(x)$$

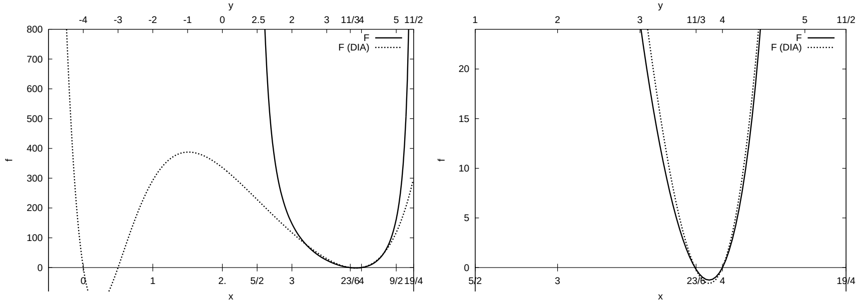
$F(x)$ is defined for $\frac{5}{2} < x < \frac{19}{4}$

$F \to \frac{5\times5\times2^{-3}\times\pi^3}{x-\frac{5}{2}}, \quad x \to \frac{5}{2}$ and $F \to \frac{5\times11\times19\times2^{-13}\times\pi^3}{\frac{19}{4}-x} \approx, \quad x \to \frac{19}{4}$





Comparison of the F function with that of DIA method



Kolmogorov type spectra

Kolmogorov type spectra

$$N_k^{(1)} = c_p \left(\frac{P_0}{g^2}\right)^{1/3} \frac{1}{k^4},$$

$$N_k^{(2)} = c_q \left(\frac{Q_0}{g^{3/2}}\right)^{1/3} \frac{1}{k^{23/6}}.$$

$$c_p = \left(\frac{3}{2\pi\, F'(4)}\right)^{1/3}, \quad c_q = \left(\frac{3}{2\pi\, |F'(23/6)|}\right)^{1/3}$$

Existence of the "window of opportunity" $\frac{5}{2} < x < \frac{19}{4}$ is the result of cancellation of the terms

$$S_{\rm nl} = F_k - \Gamma_k N_k$$

However, each particular term typically is divergent at small wave numbers. Now for $k \gg k_1, k_3$

$$\Gamma_k = 2\pi g^2 \int |\, T_{kk_1,kk_3}|^2 \, \delta(\omega_{k_1} - \omega_{k_3}) \, N_{k_1} N_{k_3} \, dk_1 dk_2$$

For a spectrum narrow in angle

$$\Gamma_k = 8\pi g^{3/2} \, k^2 \, \cos^2\theta \int_0^\infty k_1^{13/2} \, \tilde{N}^2(k_1) \, dk_1$$

For the "mature sea" the Pierson-Moskowitz spectrum

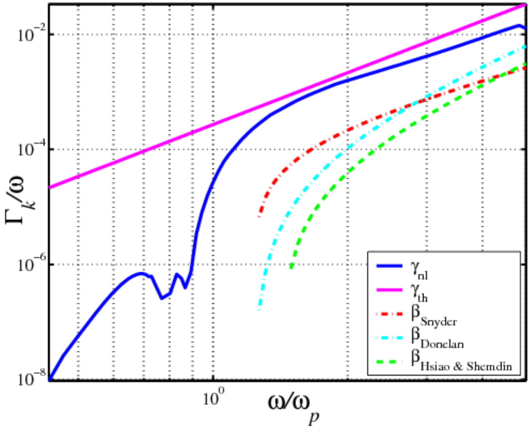
$$N_k = \frac{3}{2}\,\frac{E}{\sqrt{g}}\,\frac{k_p^{3/2}}{k^4}\theta(k - k_p)$$

$$\Gamma_k = 36\pi\omega\left(\frac{\omega}{\omega_p}\right)^3 \mu_p^4 \cos^2\theta$$

$$\mu_p^2 = \frac{E\omega^4}{g^2}$$

$$\Gamma_k = 7.06 \cdot 10^{-4}\omega\left(\frac{\omega}{\omega_p}\right)^3 \cos^2\theta$$

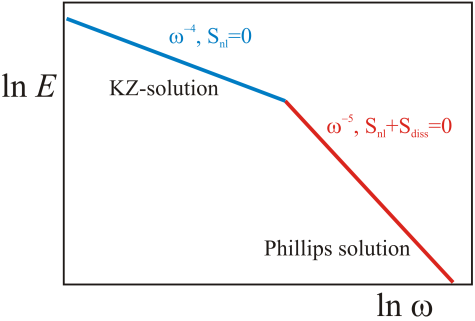
Compare nonlinear damping decrement and wind input increment



S_{nl} surpasses S_{in} and S_{diss} in order of magnitude !

Phillips-like dissipation term

One can adjust the Kolmogorov-Zakharov and dissipative (Phillips) solutions – both are self-similar !



$S_{diss} = -\lambda_{Phillips}\omega\Theta(\omega/\omega_p - q)\mu_w^4 E(\omega)$, where $\mu_w^2 = E\omega^5/g^2$

$q \simeq 3 \div 4$ – dissipation cut-off

Dissipation term

$$S_{diss} = -\lambda_{Phillips}\omega\Theta(\omega/\omega_p - q)\mu_w^4 E(\omega)$$

has the same homogeneity properties as the collision integral

$$S_{nl}(aE,\ b\omega) = a^3 b^{11} S_{nl}(E,\omega)$$

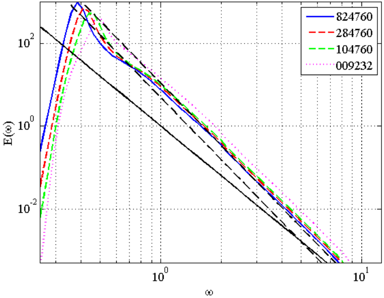
For power-like spectra $E \sim \omega^{-z}$

$$S_{nl} = C(z)\omega\Theta(\omega/\omega_p - q)\mu_w^4 E(\omega)$$

Spectral slope -5 fixes the dissipation rate

$$\lambda_{Phillips} = C(5) = 2.19$$

Domains of constant and decaying fluxes are co-existing. The Phillips tail close to ω^{-5} is seen well in 1 day of evolution.



Phillips' spectra and fluxes for $\lambda_{Phillips} = 1.22$
Dashed line extreme Phillips' constants $\alpha_p = 0.0081$ and $\alpha_p = 0.018$ (see *Dynamics and Modelling of Ocean Waves by Komen et al. 1994*). Solid line $E(\omega) \sim \omega^{-4}$.

Self-similar solutions for Hasselmann's equation

Equation written for $\varepsilon(\omega)$ (where $\varepsilon(\omega)\,d\omega = \omega(k)N(k)k\,dk$) has a family of self-similar solutions

$$\varepsilon(\omega, t) = t^{p+q}\varepsilon_0(pt^q)$$

p and q are related as

$$q = \frac{2p+1}{9}$$

The new form of S_{diss} may be included here without violation of self-similarity!

For swell

$$q = \frac{1}{11}, \quad p = -\frac{1}{11}, \quad p = -q$$

$$N = \int \frac{\varepsilon(\omega)}{\omega}\,d\omega = \mathrm{const}$$

For $N \sim t$

$$q = \frac{3}{11}, \quad p = \frac{8}{11}$$

Let $\mu^2 = \dfrac{\varepsilon\omega^4}{g^2}$, $\nu = \omega_p t$.
The universal quantity

$$\alpha_0^3 = \mu_p^4 \nu$$

does not depend on choice of p !
(See the talk of Badulin and all)
The stationary equation

$$\frac{\cos\theta}{\omega}\frac{\partial N}{\partial x} = S_{\mathrm{nl}}$$

has similar solution with

$$q = \frac{2p+1}{10}$$

(See the talk of Pushkarev and all)

Quadruplet form of the Hasselmann's equation

$$\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3 + \mathbf{k}_4$$

$$\omega_1 + \omega_2 = \omega_3 + \omega_4$$

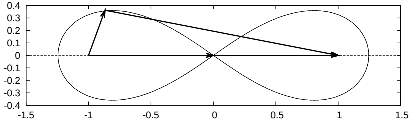
Let us study and classify the quadruplets satisfying the resonance conditions.
Qudruplets may be scaled and rotated. Base vector $\mathbf{k}_B$ defines the quadruplet scale and direction

$$\mathbf{k}_B = \frac{1}{2}\left(\mathbf{k}_1 + \mathbf{k}_2\right) = \frac{1}{2}\left(\mathbf{k}_3 + \mathbf{k}_4\right)$$

The dimensionless modulus s

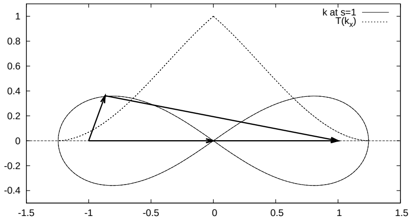
$$s = \frac{1}{2}\,\frac{\omega_1 + \omega_2}{\sqrt{g\mathbf{k}_B}} = \frac{1}{2}\,\frac{\omega_3 + \omega_4}{\sqrt{g\mathbf{k}_B}}$$

1. Central (DIA-like) quadruplets. Reside at the Phillips curve



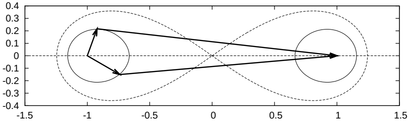
$s \approx 1$, $k_1 \approx 1$, $k_2 \approx 1$
They transfer most of energy.

T coefficient for central quadruplets (not to scale)



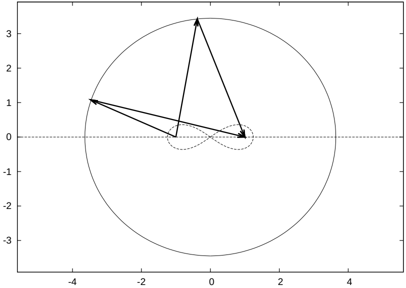
T vanish at Phillips curve sides. The wave interactions from these quadruplets are insignificant.

2. Long-short interactions



$s \to \frac{\sqrt{2}}{2}$, $k_1 \ll k_2$, $k_3 \ll k_4$
Most special kind of quadruplets.

3. Angular quadruplets



$s \to \infty,\ k_1 \approx -k_2,\ k_3 \approx -k_4,\ k_1 \approx k_3$
They play significant role in transfering the energy to the oblique directions

We need two more parameters for quadruplets. We introduce dimensionless λ and μ

$$\tilde{\omega}_1 = s(1-\lambda)$$

$$\tilde{\omega}_2 = s(1+\lambda)$$

$$\tilde{\omega}_3 = s(1-\mu)$$

$$\tilde{\omega}_4 = s(1+\mu)$$

(The tilde signifies dimensionless variables.)

Any quadruplet parameter may be calculated from s, λ and μ using only elementary functions. For example:

$$\cos\theta_1 = \frac{1-2s^4(\lambda+\lambda^3)}{s^2(1-\lambda)^2}$$

$$\cos\theta_3 = \frac{1-2s^4(\mu+\mu^3)}{s^2(1-\mu)^2}$$

The Hasselmann equation may be rewritten in dimensionless quadruplet variables:

$$\frac{\partial N_{\mathbf{k}}}{\partial t} = \pi g^{3/2} k^{19/2} \int\limits_{\tilde{\mathbf{k}}_1, \tilde{\mathbf{k}}_3} \left(\tilde{T}_{\tilde{\mathbf{k}}_1 \tilde{\mathbf{k}}_2 \tilde{\mathbf{k}}_3 \tilde{\mathbf{k}}_4} \right)^2 \tilde{k}_1^{-\frac{23}{2}} \times$$

$$\times \left(N N_3 N_4 + N_2 N_3 N_4 - N N_2 N_3 - N N_2 N_4 \right) \delta(s - s')\, d\tilde{\mathbf{k}}_1\, d\tilde{\mathbf{k}}_3$$

The base vector k_B is impicitly present inside the integral.
It is calculated from $k = \tilde{k}_1 k_B$.
s' is calculated from k_3 and k_4.

The energy spectrum ε is introduced in such a way that

$$\varepsilon\, d\omega\, d\theta = \omega N\, d\mathbf{k}$$

$$\varepsilon = \frac{2\omega^4}{g^2} N$$

The Hasselmann equation for energy is

$$\frac{\partial \varepsilon}{\partial t} = \frac{\pi}{4}\, g^{-4} \int\limits_{\tilde{\mathbf{k}}_1, \tilde{\mathbf{k}}_3} \tilde{T}^2\, \omega_B^{11} \times$$

$$\times \left(\frac{\varepsilon \varepsilon_3 \varepsilon_4}{\tilde{\omega}_1^4 \tilde{\omega}_3^4 \tilde{\omega}_4^4} + \frac{\varepsilon_2 \varepsilon_3 \varepsilon_4}{\tilde{\omega}_2^4 \tilde{\omega}_3^3 \tilde{\omega}_4^4} - \frac{\varepsilon \varepsilon_2 \varepsilon_3}{\tilde{\omega}_1^4 \tilde{\omega}_2^4 \tilde{\omega}_3^4} - \frac{\varepsilon \varepsilon_2 \varepsilon_4}{\tilde{\omega}_1^4 \tilde{\omega}_2^3 \tilde{\omega}_4^4} \right) \delta(s - s')\, d\tilde{\mathbf{k}}_1\, d\tilde{\mathbf{k}}_3$$

$$\varepsilon_i = \varepsilon(\omega_i, \theta_i) = \varepsilon(\,\tilde{\omega}_i \omega_B,\, \tilde{\theta}_i + \theta_B)$$

($\tilde{\theta}_i$ is the angle between $\mathbf{k}_B$ and $\mathbf{k}_i$)
Let q mean the quadruplet $\tilde{\mathbf{k}}_1$, $\tilde{\mathbf{k}}_2$, $\tilde{\mathbf{k}}_3$, $\tilde{\mathbf{k}}_4$.

$$S(\omega_B, \theta_B, q) = \omega_B^{11} \left(\frac{\varepsilon_1 \varepsilon_3 \varepsilon_4}{\tilde{\omega}_1^4 \tilde{\omega}_3^4 \tilde{\omega}_4^4} + \frac{\varepsilon_2 \varepsilon_3 \varepsilon_4}{\tilde{\omega}_2^4 \tilde{\omega}_3^3 \tilde{\omega}_4^4} - \frac{\varepsilon_1 \varepsilon_2 \varepsilon_3}{\tilde{\omega}_1^4 \tilde{\omega}_2^4 \tilde{\omega}_3^4} - \frac{\varepsilon_1 \varepsilon_2 \varepsilon_4}{\tilde{\omega}_1^4 \tilde{\omega}_2^3 \tilde{\omega}_4^4} \right)$$

The quadruplet form of the kinetic equation is

$$\frac{\partial \varepsilon(\omega, \theta)}{\partial t} = \frac{\pi}{4}\, g^{-4} \int\limits_{\tilde{\mathbf{k}}_1 < \tilde{\mathbf{k}}_2,\, \tilde{\mathbf{k}}_3 < \tilde{\mathbf{k}}_4} \tilde{T}^2 \, \Big(S(\,\omega/\tilde{\omega}_1,\, \theta - \tilde{\theta}_1,\, q) + S(\,\omega/\tilde{\omega}_2,\, \theta - \tilde{\theta}_2,\, q) -$$

$$- S(\,\omega/\tilde{\omega}_3,\, \theta - \tilde{\theta}_3,\, q) - S(\,\omega/\tilde{\omega}_4,\, \theta - \tilde{\theta}_4,\, q) \Big) \, \delta(s - s')\, d\tilde{\mathbf{k}}_1 \, d\tilde{\mathbf{k}}_3$$

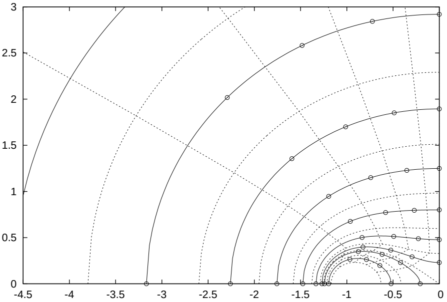
Instead of $\delta(s - s')\, d\tilde{\mathbf{k}}_1\, d\tilde{\mathbf{k}}_3$ we may use any other parametrisation.

$$\delta(s - s')\, d\tilde{\mathbf{k}}_1\, d\tilde{\mathbf{k}}_3 = J\, ds\, d\lambda\, d\mu$$

$$J = \frac{64 s^6 (1 - \lambda^2)(1 - \mu^2)}{\sin(\theta_1 - \theta_2)\sin(\theta_3 - \theta_4)}$$

New numerical algorithm for solving the Hasselmann's equation

The algorithm is based on the quadruplet grid. First the set of modulae is chosen than set of grid points for each modulus is chosen. Each point gives two quadruplet vectors, thus a quadruplet consists of two grid points.
The figure shows set of 10 modulae from $s = 0.96$ to $s = 2.5$ with 8 points for each modulus. This is nearly minimal grid adequate for the calculations.



The k plane is divided into cells each containing a grid points. Cell areas are directly substituted into calculations thus avoiding the use of the Jacobian.
Most significant grid points are those near the center of the Phillips curve. Any incorrectness in cell size or the matrix coefficient yields significant discrepancy in the energy flux.

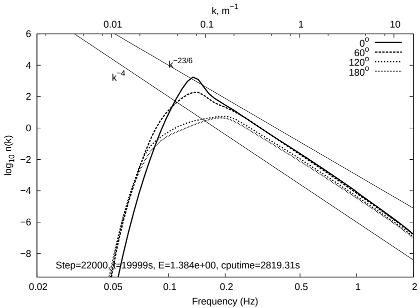
Generally, spectrum is steep in k but not by θ. We choose the grid points to be equidistant by k to better cover the steepness of k.

Comparison with Resio–Tracy and DIA methods

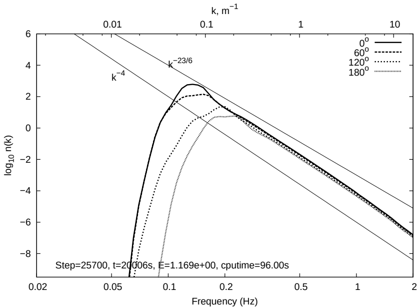
Test calculation results:
Pumping at $f_0 < f < f_{\max}$ and $\cos\theta > 0$

$$\gamma = 1.6 \times 10^{-4}\, \frac{f}{2\pi}(f - f_0)^2 \cos\theta$$

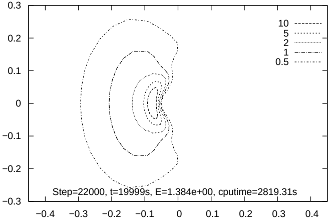
$f_0 = 0.1\mathrm{Hz}$ ($f_{\max} = 1\mathrm{Hz}$ is introduced for techical reasons)
No damping is required. The energy is naturally transfered by the Kolmogorov cascade outside of the computation zone.



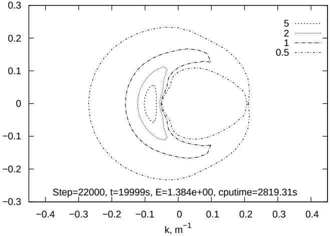
New calculation.



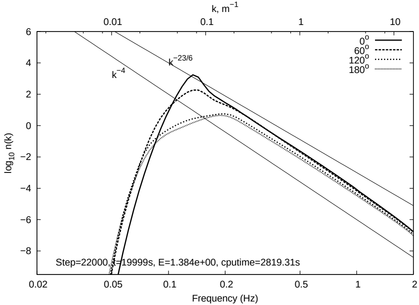
DIA calculation

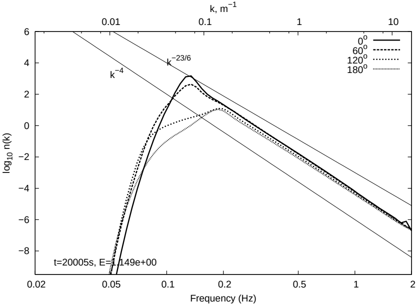


New calculation.



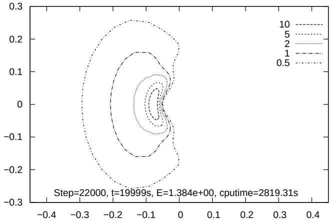
DIA calculation



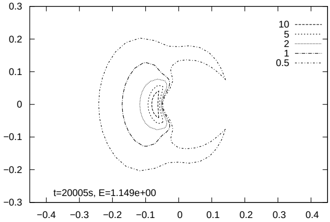


New calculation. Resio–Tracy

Algorithm works several times faster than WRT (Wenn-Resio-Tracy) one and 20 times slower than DIA. There are chances to make it faster. 2400 quadruplets are used in the calculations in figures; we plan to make the namber much smaller.



New calculation.



Resio–Tracy

Summary

The Kolmogorov-Zakharov soluti
inverse cascades) are universal fe
Hasselmann equation;