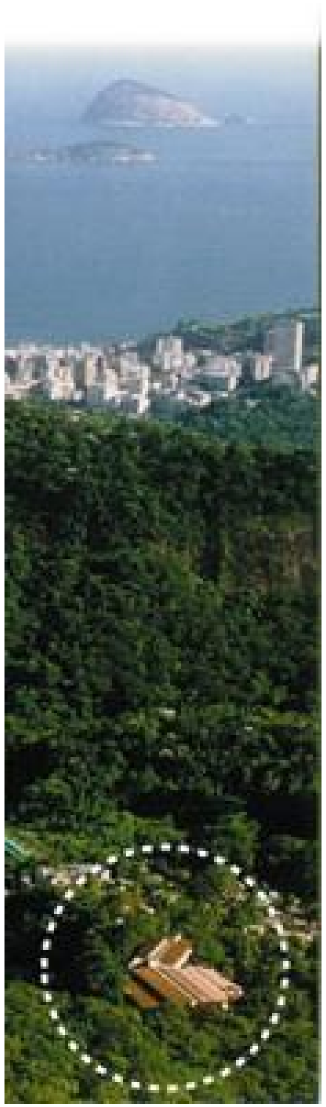

Blowup as a driving mechanism of turbulence in shell models

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Hydrodynamic turbulence

Navier–Stokes equations $$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p + \nu \Delta \mathbf{v} + \mathbf{f}, \quad \nabla \cdot \mathbf{v} = 0$$

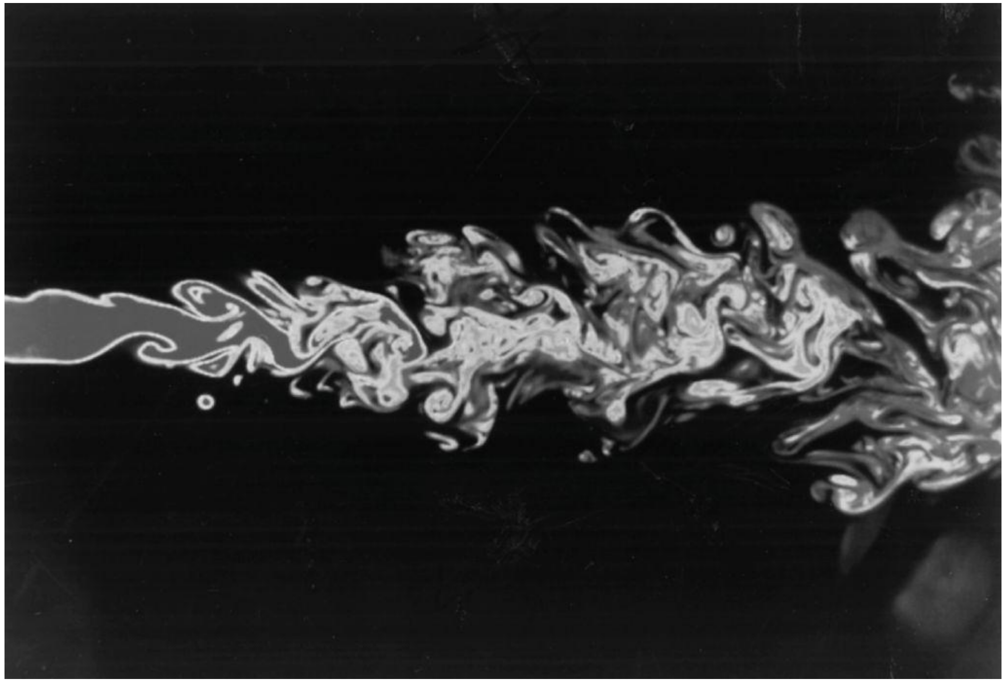
Large Reynolds number

$$\mathrm{Re} = \frac{VL}{\nu} >> 1$$

$$\left(10^{>3}\right)$$

Fully developed turbulence

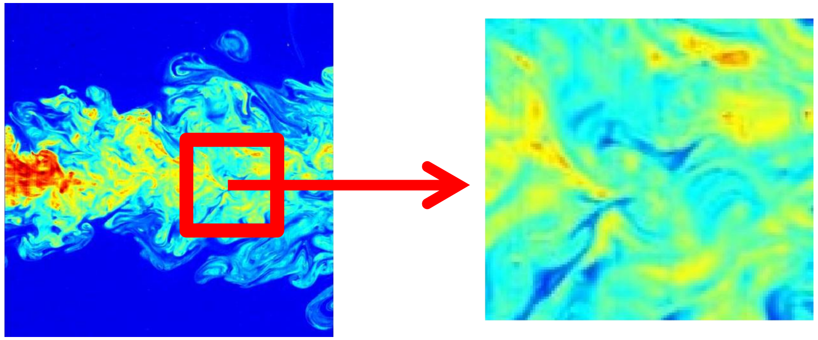
$$\mathrm{Re} \to \infty$$



Open problems:
Existence and uniqueness of smooth solution (with and without viscosity), explanation of turbulent statistics, dissipation anomaly etc.

Kolmogorov’s theory (1941) and anomaly

turbulent statistics at small scales: universal, isotropic, homogeneous



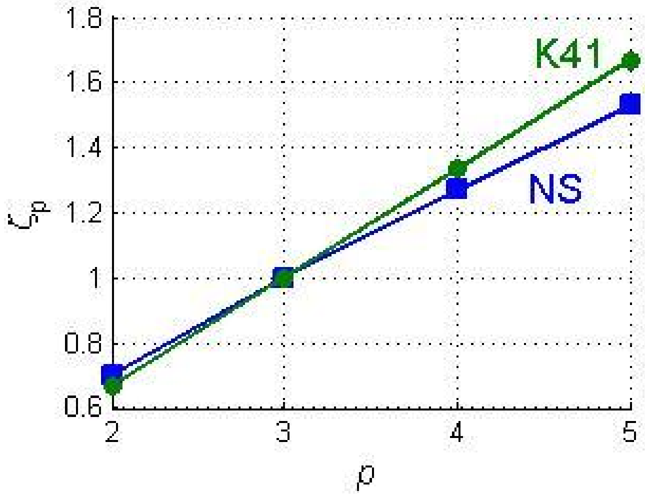
Velocity moments in inertial range):

$$S_p(r) = \left\langle |\, \delta \mathbf{v}\, |^p \right\rangle \propto r^{p/3}$$

(Kolmogorov - K41)

Experiment: $S_p(r) \propto r^{\zeta_p},\ \zeta_p \neq p/3$

(comment of Landau, 1941)

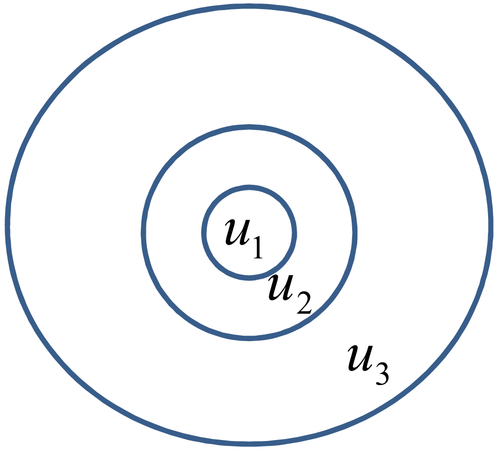


Dissipation anomaly: positive limit of dissipation rate as $\mathrm{Re} \to \infty$ (Onsager,1946)

Singular (1/3-Hölder) velocity field (Onsager 1946) and blowup problem (open)

Interpretation with multifractal model (Parisi&Frisch 1985)

Shell models of turbulence



Descrete variables: $\mathbf{k} \to k_n = 2^n, \quad r_n = 2^{-n}, \quad \mathbf{v} \to u_n \in \mathrm{C}$

Sabra model

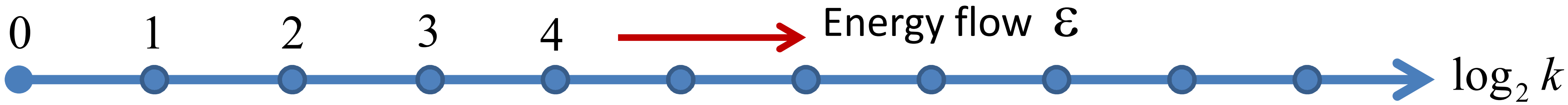
(L'vov, Podivilov, Pomyalov, Procaccia, Vandembroucq, 1998),
modified Gledzer-Ohkitani-Yamada (GOY) model, (1973-89)

$$\frac{du_n}{dt} = i(k_{n+1}u_{n+2}u^*_{n+1} - \tfrac{1}{2}k_n u_{n+1}u^*_{n-1} + \tfrac{1}{2}k_{n-1}u_{n-1}u_{n-2}) - \nu k_n^2 u_n + f_n$$

$$k_n = 2^n$$

$$r_n = 2^{-n}$$

(quadratic nonlinearity, conservation of energy and helicity, viscousity etc.)



Forcing range

Inertial range

Viscous range

$$\frac{dU_n}{dt} = -\tfrac{1}{4}U_{n+2}U^*_{n+1} + \tfrac{1}{2}U_{n+1}U^*_{n-1} + 2U_{n-1}U_{n-2}, \quad U_n = ik_n u_n$$

Structure functions:

$$S_p(k_n) = \langle |u_n|^p \rangle \propto k_n^{-\varsigma_p}$$

Anomalous scaling:

$Sabra: \ \varsigma_2 = 0.72, \ \varsigma_3 = 1, \ \varsigma_4 = 1.26, \ \varsigma_5 = 1.49$

$NS: \ \varsigma_2 = 0.7, \ \varsigma_3 = 1, \ \varsigma_4 = 1.27, \ \varsigma_5 = 1.53$

Blow-up in shell model

Viscous model: global existence and uniqueness for $(U_n) \in \ell^2$

(Constantin, Levant, Titi, 2007)

Inviscid model: finite time blowup

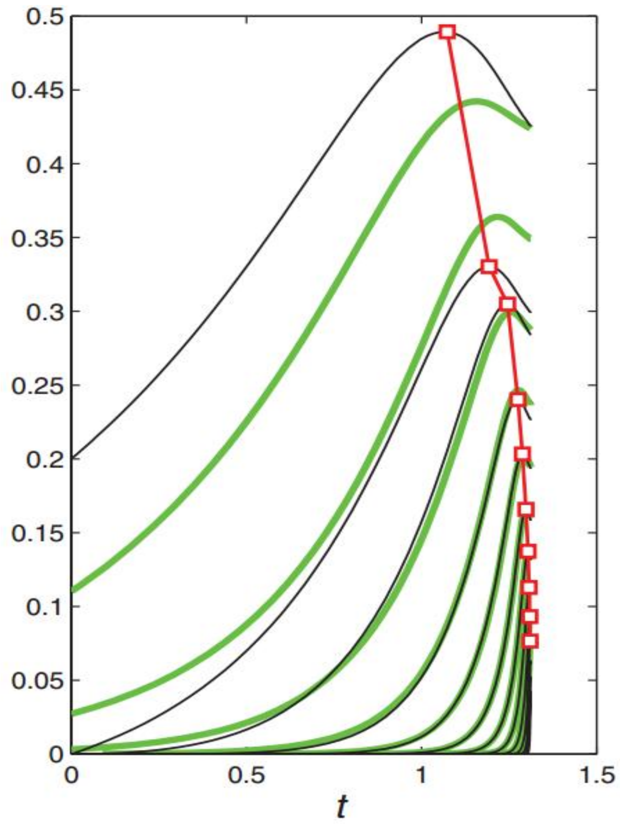
(Dombre&Gilson 1998; AM 2012, 2013)

Self-similar universal blowup structure
(when only large scales are perturbed in I.C.)

$$u_n(t) = -iu_* k_n^{-y_0} f[u_*(t_* - t) k_n^{1-y_0}]$$

$$y_0 = 0.281$$

$$u_n \propto k_n^{-y_0}, \quad t_* - t \propto k_n^{1-y_0}$$



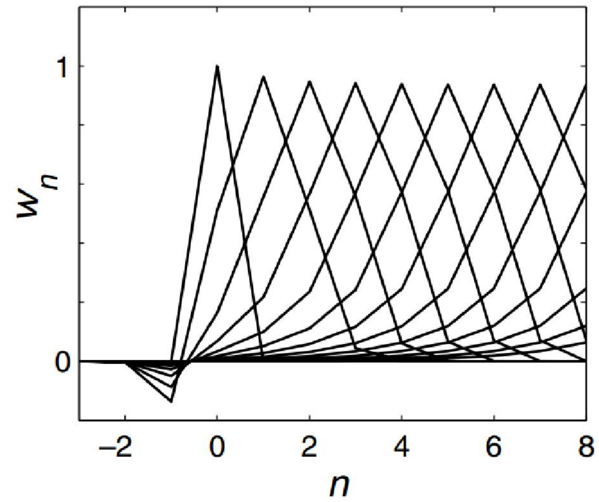
Blow-up theory (inviscid model)

$$\frac{du_n}{dt} = i(k_{n+1}u_{n+2}u_{n+1}^* - \tfrac{1}{2}k_n u_{n+1}u_{n-1}^* + \tfrac{1}{2}k_{n-1}u_{n-1}u_{n-2})$$

Renormalized variables (Dombre&Gilson 1998)

$$t = t_0 + \int_0^{\tau} \exp\left[-\int_0^{\tau'} A(\tau'')d\tau''\right]d\tau', \quad u_n = -ik_n^{-1}\exp\left[-\int_0^{\tau} A(\tau')d\tau'\right]w_n$$

and renormalized system



$$\frac{dw_n}{d\tau} = N_n[w] - Aw_n$$

$$N_n[w] = -\tfrac{1}{4}w_{n+2}w_{n+1}^* + \tfrac{1}{2}w_{n+1}w_{n-1}^* + 2w_{n-1}w_{n-2}$$

$$A = \mathrm{Re}\sum_n w_n^* N_n[w] \Big/ \sum_n |w_n|^2 \quad \Rightarrow \quad \sum_n |w_n|^2 = const$$

Existence of solution for all times (no blowup) if $\sum_n |w_n|^2$ is finite (AM 2013)

Fixed-point attractor of Poincaré map leads to Dombre&Gilson traveling wave solution, which in turn implies universal self-similarity (AM 2013)

Blowup in other shell models: literature review

Numerical evidence of blowup with self-similar structure:

Siggia 1978
Nakano 1988
Uhlig & Eggers 1997
L’vov, Pomyalov, Procaccia 2001; L’vov 2001

Rigorous results for Desniansky-Novikov shell models (no intermittency):

Katz & Pavlovic 2005
Kiselev & Zlatos 2005
Cheskidov, Friedlander, Pavlovic 2007

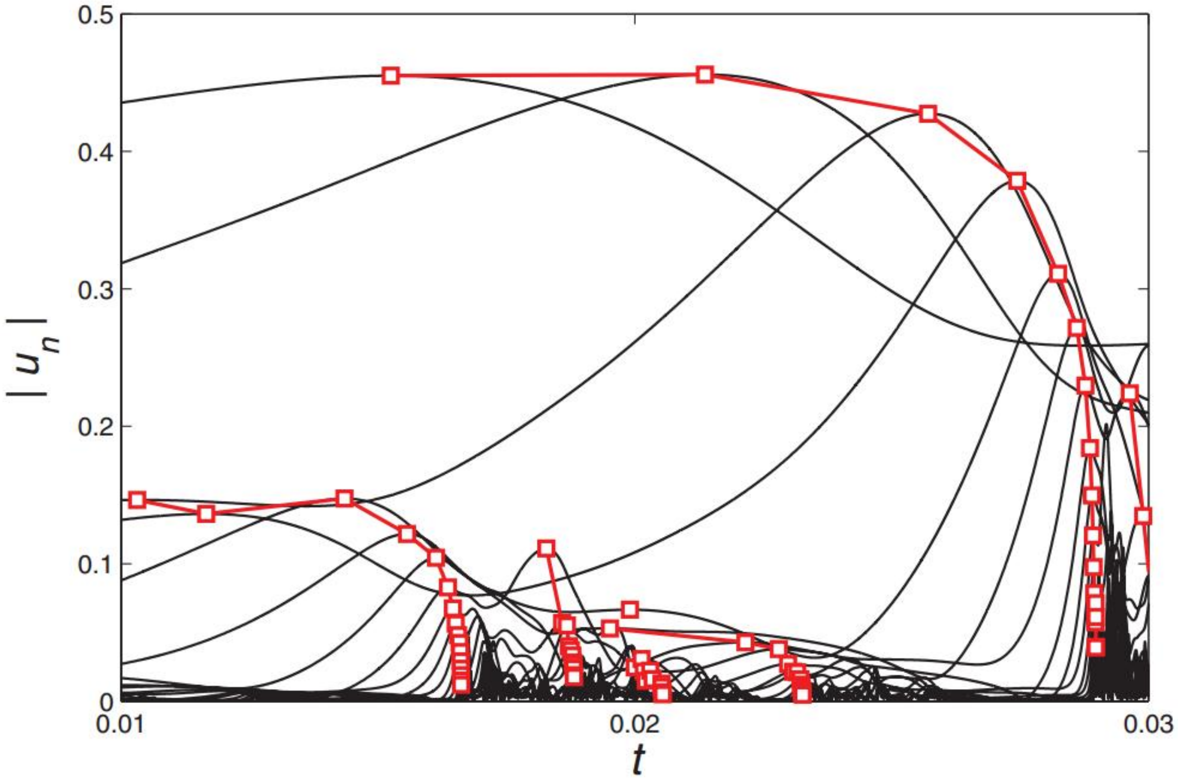
Renormalization method:

Dombre & Gilson 1998
AM 2012, 2013

Instantons

Simulations for 40 shells
with small viscosity, Re ~ 10^{14}

If small viscosity is introduced,
blowup phenomena reduce to
instantons:
extreme events correlated in
space and time



Instantons are identified using local maximums of shell velocities:

$$v_n = \max_t u_n(t)$$

Large part of maxima ~ 80% belong to instantons reaching the viscous range!

Anomalous scaling in terms of instantons

Viscosity moments: $S_p(k_n) = \langle |u_n|^p \rangle \propto k_n^{-\varsigma_p}$ $\Delta t_n \approx (k_n v_n)^{-1}$

$$\langle |u_n|^p \rangle = \lim_{T\to\infty} \frac{1}{T}\int_0^T |u_n|^p \, dt \approx \lim_{T\to\infty} \frac{1}{T} \sum_{\text{all instantons}} v_n^p \Delta t_n \approx \lim_{T\to\infty} \frac{1}{Tk_n} \sum_{\text{all instantons}} v_n^{p-1}$$

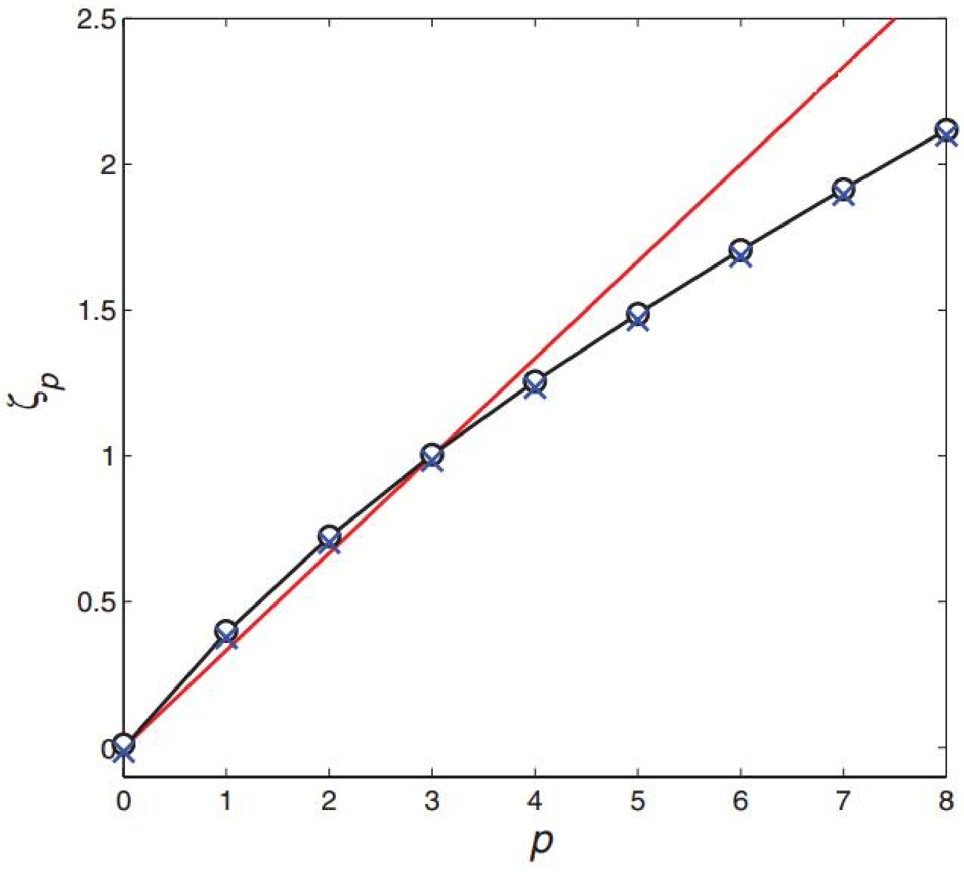
Viscosity moments in terms of instanton amplitudes:

$$S'_p(k_n) = \lim_{T\to\infty} \frac{1}{Tk_n} \sum_{\text{all instantons}} v_n^{p-1}$$

Inertial range scaling

$$S'_p(k_n) \propto k_n^{-\zeta_p}$$

Same values of anomalous scaling exponents!



Some new interpretations of scaling exponents

Instantons are dense in space-time (n,t)

$$\zeta_0 = 0 \quad \Rightarrow \quad \left\langle |u_n|^0 \right\rangle \approx \lim_{T\to\infty} \frac{1}{T} \sum_{\text{all instantons}} \Delta t_n \propto const$$

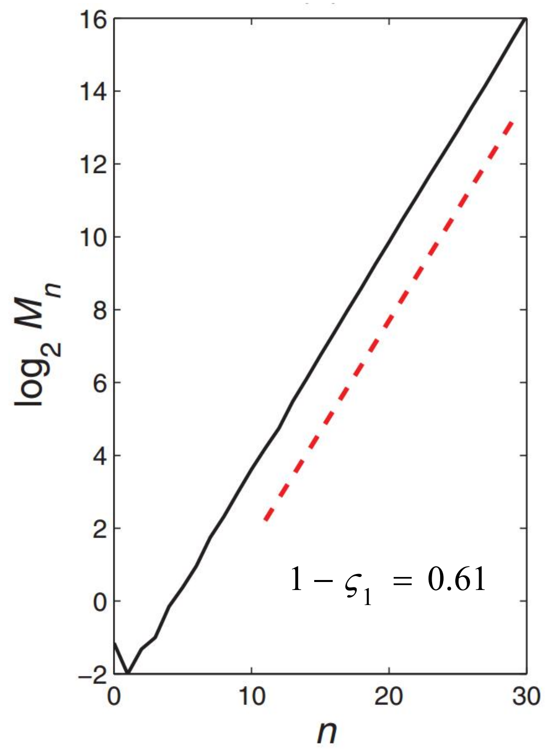
(instanton lifetime is $\Delta t_n \approx (k_n v_n)^{-1}$)

Instanton creation rate

$$\zeta_1 = 0.39 \quad \Rightarrow$$

$$\left\langle |u_n|^1 \right\rangle \approx \lim_{T\to\infty} \frac{1}{Tk_n} \sum_{\text{all instantons}} 1 \propto k_n^{-\varsigma_1} \quad \Rightarrow$$

$$\text{instanton creation rate} \propto k_n^{1-\varsigma_1}$$

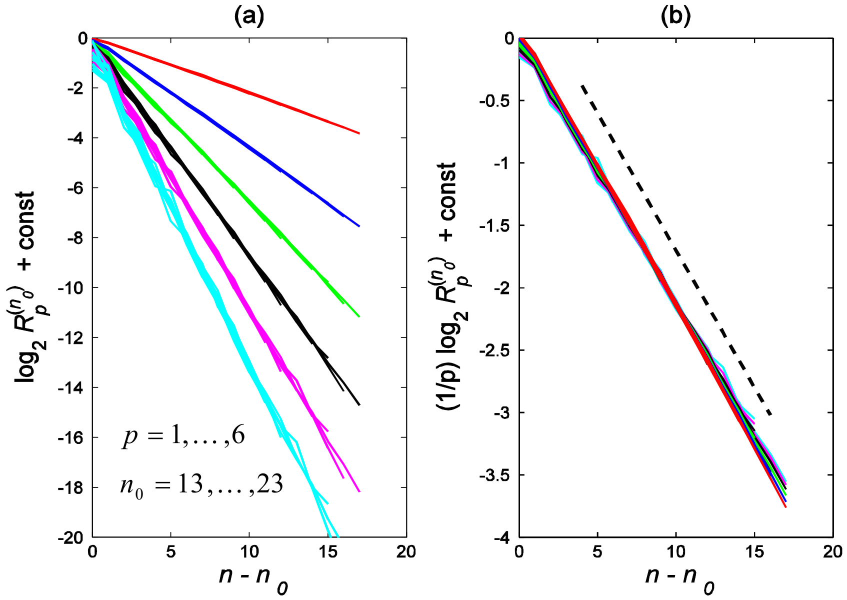


Self-similar statistics of the instanton

Instanton structure functions:

$$R_p^{(n_0)}(k_n) = \lim_{T\to\infty} \frac{1}{T} \sum_{\substack{\text{all instantons}\\ \text{created in shell } n_0}} v_n^p,$$

$$R_p^{(n_0)}(k_n) \propto k_n^{-yp},$$

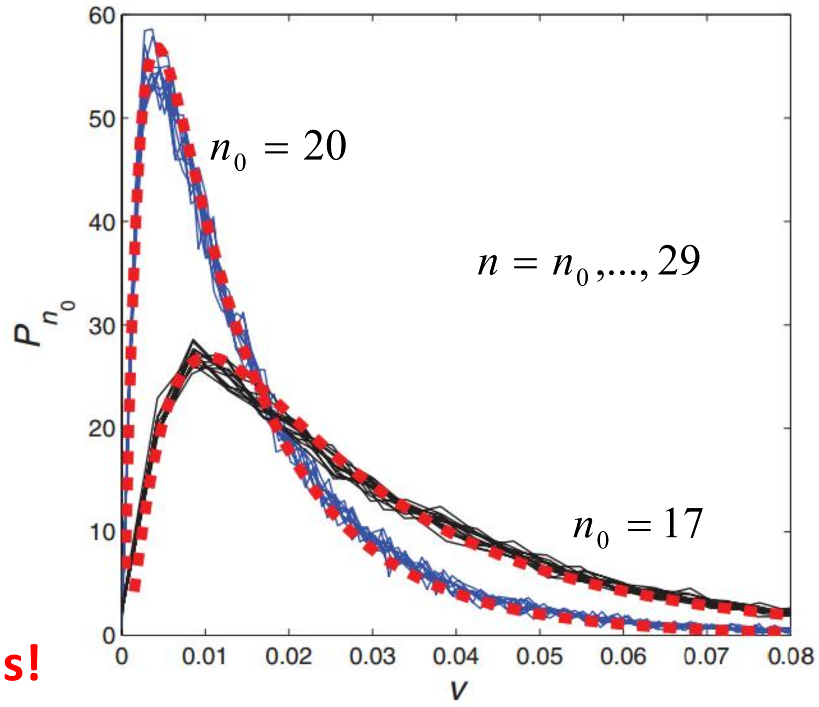


Exponent: y = 0.225 (blowup value 0.281)

Self-similarity of probability density function:

$$P_{n_0}(v) = 2^{-y\Delta n} P_{n,n_0}\left(2^{-y\Delta n} v\right)$$

$P_{n,n_0}(v)\,dv$ is the probability to find the maximum v in shell n for the instanton created in shell n_0



No anomaly of scaling exponents for instantons!

Large deviation principle: derivation

Scaling rule for the moments

$$S'_p(k_n) = k_n^{-1} \sum_{n_0=0}^{n} R_{p-1}^{(n_0)}(k_n) = k_n^{-1} \sum_{n_0=0}^{n} R_{p-1}^{(n_0)}(k_{n_0}) 2^{-(p-1)y(n-n_0)} \propto k_n^{-\varsigma_p}$$

$$\Rightarrow \quad R_{p-1}^{(n)}(k_n) \propto k_n^{1-\zeta_p} \quad \Rightarrow \quad \frac{1}{T} \sum_{\substack{\text{for instantons} \\ \text{born at shell } n}} v_n^{p-1} = \boxed{M_n \int v^{p-1} P_n(v) dv \propto k_n^{1-\zeta_p}}$$

Change of variables

$$v \mapsto a = \frac{1}{n} \log_2 \frac{v}{v_*}, \quad P_n(v) \mapsto \rho_n(a) = \rho_* n M_n P_n(v)$$

$$\boxed{\int 2^{npa} \rho_n(a) da \propto 2^{n(1-\zeta_p)}}$$

Solving using Gärtner-Ellis theorem

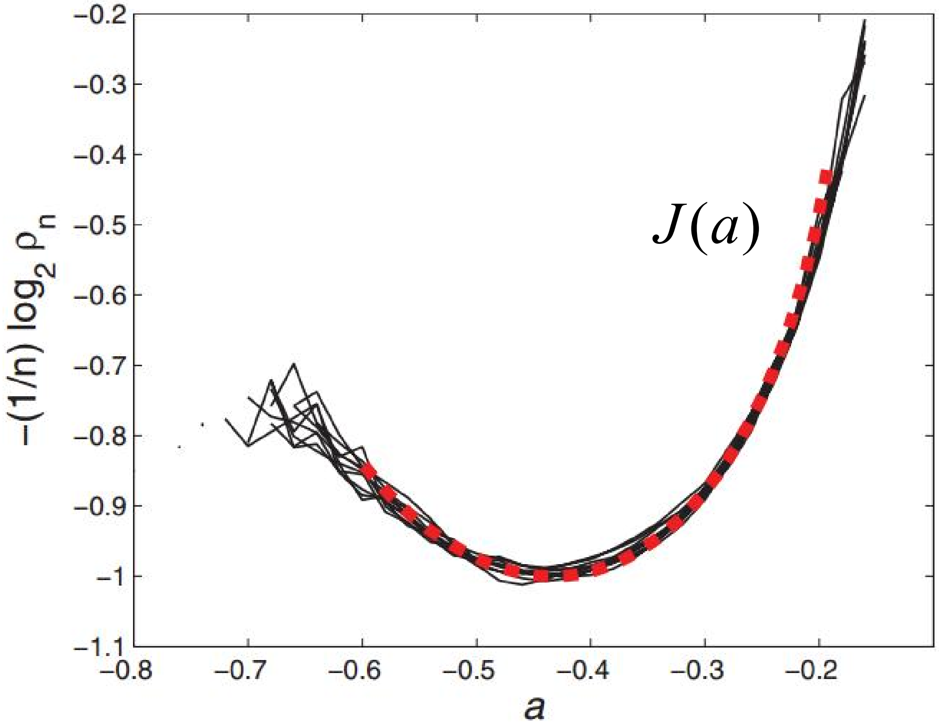
$$\rho_n(a) \propto 2^{-nJ(a)} = k_n^{-J(a)} \qquad J(a) = pa - (1-\zeta_p), \quad a = -\frac{d\zeta_p}{dp}$$

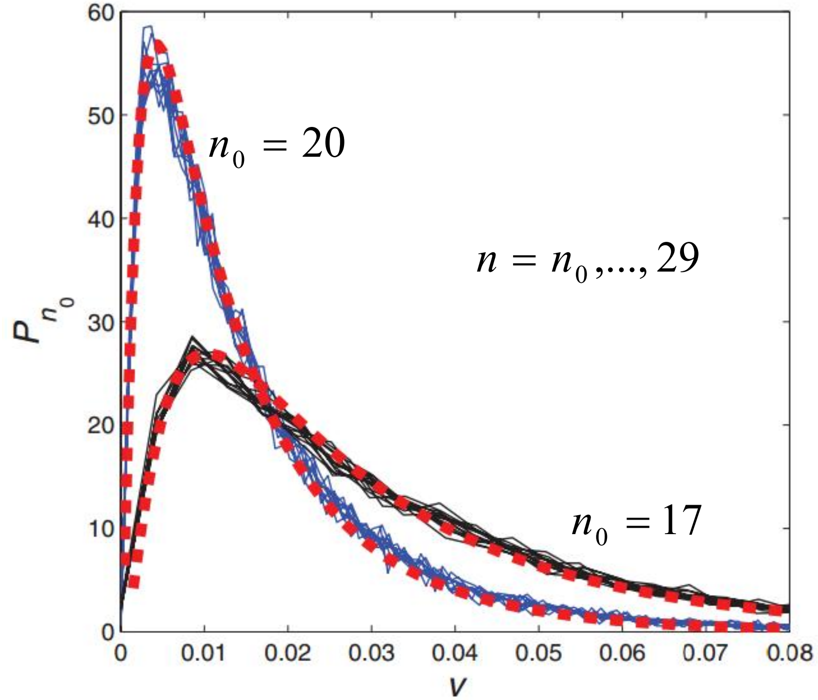
rate function

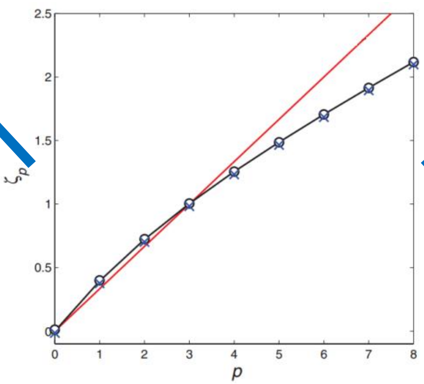
Large deviation principle: numerical validation

$$\rho_n(a) \propto 2^{-nJ(a)} = k_n^{-J(a)}$$

$$M_n P_n(v) \propto k_n^{-J(a)}, \quad a = \frac{1}{n}\log_2 \frac{v}{v_*}$$







Instanton creation (simple model)

Instanton of amplitude $v_n = 1$ creates

$$\varphi(v)dv$$

instantons of amplitude v in the same shell.
Rescaled value for $v_n = v'$:

$$\frac{1}{v'}\varphi\left(\frac{v}{v'}\right)dv$$

Direct computation of PDFs for instantons (using self-similarity) and of velocity moments yields the explicit anomalous exponent

$$\zeta_p = 1 + (p-1)y - \log_2(1 + \varphi_{p-1}), \qquad \varphi_p = \int_0^\infty \xi^p \varphi(\xi) d\xi$$

Contribution of self-similarity of instantons

Contribution of instanton creation mechanism

Conclusions

Blowup in is linked to an attractor of the renormalized system. Such an attractor (soliton) determines universal asymptotic properties of the blowup, both for scaling and shape.

Instantons are **statistical** objects originating from blowup. They are **universal** and **self-similar** with scaling properties influenced by collective turbulent behavior: $0.281 \to 0.225$.

Large deviation principle allows analytic relation of instanton statistics (PDFs) to anomalous scaling exponents. Excellent agreement with numerical simulations.

Anomaly of scaling exponents for a shell model results from the process of **instanton creation**.

References: PRE 85 066317 (2012); PRE 86 025301 (R) (2012);
PRE 87 053011 (2013); Nonlinearity 26 1105 (2013)

Blowup: scaling vs. creation in turbulence statistics

Exponents for a single blowup: $\zeta_p = 1 + (p-1) y_0$

Exponents for a "gas" of instantons: $\zeta_p = 1 + (p-1) y - \Delta_p$