Projective differential geometry of multidimensional dispersionless integrable hierarchies

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Outline

Lax pairs in terms of vector fields (diff. operators of the first order)
*V.E Zakharov and A.B. Shabat, Integration of nonlinear equations of mathematical physics by the method of inverse scattering problem. II, Func. Anal. and Appl.(in Russian),**13**(3), 13–22 (1979)*

1. General technique
   In collaboration with B.G. Konopelchenko
   - Around Frobenius theorem. Integrable distributions, closed Plücker forms
   - Introducing the spectral variable.
   - How to solve? Nonlinear vector Riemann problem (or $\bar{\partial}$ problem)
   - Polynomial case. Examples of dispersionless integrable systems
2. Example: Doubrov-Ferapontov general heavenly equation
   The work in progress

Equivalent descriptions

The following objects are in one-to-one correspondence:

- Involutive distributions
- Gauge-invariantly closed Plücker forms

Integrable distributions

Local coordinates $\mathbf{x} = (x_0, x_1, \ldots, x_N)$.

- *Distribution*: k-dimensional subspace of the tangent space $\Delta_x \subset T_x$, depending smoothly on $\mathbf{x}$ (there exists a basis of smooth vector fields).
- *Involutive distribution*: $[\Delta, \Delta] \subset \Delta$
- *Frobenius theorem*: The distribution is integrable (corresponds to a foliation) $\Leftrightarrow$ the distribution is in involution

There are also dual formulations in terms of differential forms

Plücker forms

Local coordinates $\mathbf{x} = (x_0, x_1, \dots, x_N)$, $N \leqslant \infty$. The m-form

$$\Omega_m = \sum_{0 \leqslant i_0 \leq \dots \leqslant i_{N-2}} \pi_{i_0\, i_1\, \dots\, i_{m-1}}(x) dx_{i_0} \wedge dx_{i_1} \wedge \dots \wedge dx_{i_{m-1}}$$

Coefficients satisfy Plücker relations

$$\sum_{l=0}^{m} (-1)^l \pi_{i_0\, \dots\, i_{m-2}\, j_l} \pi_{j_0\, \dots\, \check{j}_l\, \dots\, j_m} = 0$$

The form is defined up to a gauge. Due to Plücker relations, it is decomposable

$$\Omega_m = \omega_0 \wedge \dots \wedge \omega_{m-1},$$

defines a vector subspace in cotangent space and a distribution as a dual object. It is also easy to construct a Plücker form for a given distribution.

Question

What property of the Plücker form corresponds to the involutivity of the distribution?

Closed Plücker forms

Closedness equations

$$\left[\frac{\partial\pi_{i_0\, i_1\, \dots i_{m-1}}}{\partial x_{i_m}}\right] = 0,$$

where the bracket $[\dots]$ means antisymmetrization over all indices. For simplicity of notations we consider a pair equations with the choice of indices $(0, 1, \dots, m-1, m)$, $(0, 1, \dots, m-1, m+1)$ introducing independent affine coordinates

$$a_{1k} = (-1)^k J^{-1} \pi_{0\, \dots\, k-1\, k+1\, \dots\, m-1\, m},$$
$$a_{2k} = (-1)^k J^{-1} \pi_{0\, \dots\, k-1\, k+1\, \dots\, m-1\, m+1},$$
$$J = \pi_{0\, 1\, \dots\, m-1}$$

where $k = 0, \dots, m-1$.

With the use of Plücker relations, equations take the form

$$\frac{\partial J}{\partial x_m} + \sum_{l=0}^{m-1} \frac{\partial (J a_{1l})}{\partial x_l} = 0, \quad \frac{\partial J}{\partial x_{m+1}} + \sum_{l=0}^{m-1} \frac{\partial (J a_{2l})}{\partial x_l} = 0,$$

$$\frac{\partial a_{1k}}{\partial x_{m+1}} - \frac{\partial a_{2k}}{\partial x_m} + \sum_{l=0}^{m-1} \left(a_{2l} \frac{\partial a_{1k}}{\partial x_l} - a_{1l} \frac{\partial a_{2k}}{\partial x_l} \right) = 0, \quad k = 0, 1, \ldots, m-1.$$

In red – gauge-invariant closedness equations. Coincide with the compatibility conditions for equations on J and are equivalent to commutativity of corresponding vector fields.

Gauge-invariant closedness can be defined as the existence of a gauge, in which the Plüker form is closed in the standard sense

Proposition

Gauge-invariant closedness of the Plücker form Ω_m is equivalent to the involutiveness of the corresponding distribution

Simple examples of closedness equations

$N = 2$, Ω_1:

$$\Omega_1 = J(dx_0 - a_{10}dx_1 - a_{20}dx_2)$$

No Plücker relations. Closedness equation

$$\frac{\partial J}{\partial x_1} + \frac{\partial (Ja_{10})}{\partial x_0} = 0, \quad \frac{\partial J}{\partial x_2} + \frac{\partial (Ja_{20})}{\partial x_0} = 0,$$
$$\frac{\partial a_{10}}{\partial x_2} - \frac{\partial a_{20}}{\partial x_1} + a_{20}\frac{\partial a_{10}}{\partial x_0} - a_{10}\frac{\partial a_{20}}{\partial x_0} = 0.$$

$N = 3$, Ω_2:

$$\begin{aligned} \Omega_2 \;=\; & J(dx_0 \wedge dx_1 - a_{11}dx_0 \wedge dx_2 - a_{21}dx_0 \wedge dx_3 + a_{10}dx_1 \wedge dx_2 + \\ & + a_{20}dx_1 \wedge dx_3 - (a_{11}a_{20} - a_{10}a_{21})dx_2 \wedge dx_3) \end{aligned}$$

Closedness equations

$$\frac{\partial J}{\partial x_2} + \sum_{m=0}^{1} \frac{\partial (J a_{1m})}{\partial x_m} = 0, \quad \frac{\partial J}{\partial x_3} + \sum_{m=0}^{1} \frac{\partial (J a_{2m})}{\partial x_m} = 0,$$

$$\frac{\partial a_{1k}}{\partial x_3} - \frac{\partial a_{2k}}{\partial x_2} + \sum_{l=0}^{1} \left(a_{2l} \frac{\partial a_{1k}}{\partial x_l} - a_{1l} \frac{\partial a_{2k}}{\partial x_l} \right) = 0, \qquad k = 0, 1.$$

Introducing the spectral variable. The hierarchy

General setting

We consider (gauge invariantly) closed Plüker form Ω_m with affine coordinates (ratios of coefficients) holomorphic with respect to $\lambda = x_0$ in some complex domain. This form defines a hierarchy in terms of commuting vector fields locally holomorphic in λ, the equations of the hierarchy are the gauge-invariant closedness equations.

More specifically, we consider the forms meromorphic in the complex plane (in the affine gauge)
This setting for m=2 can be reduced to Witham hierarchy (Krichever), for m=3 to heavenly equation hierarchy (Takasaki) and connected Dunajski equation hierarchy.
Important reductions are volume (or area) conservation corresponding to closedness in the affine gauge ($J = 1$) and HCR reduction $\Omega_m \wedge d\lambda = 0$.
Most known examples correspond to the case when there exists polynomial (or Laurent polynomial) set of affine coordinates (affine gauge).

The case of Ω_1 Non-generic, no hierarchy, not clear how to solve in general.
The form

$$\Omega_1 = J(d\lambda - (u_0 + \lambda u_1)dx_1 - (v_0 + \lambda v_1 + \lambda^2 v_2)dx_2)$$

leads (after reduction) to the Liouville equation

$$\varphi_{x_1 x_2} = e^{\varphi}.$$

With higher order second polynomial it is possible to get "higher Liouville equation"

$$\varphi_{x_1 x_1 x_2} - \varphi_{x_1}\varphi_{x_1 x_2} - \frac{3}{2}e^{2\varphi} = 0$$

(Burtsev, Zakharov, Mikhailov 1987)

The case of polynomial Ω_2
Manakov-Santini hierarchy

$$\begin{aligned}\Omega_2 &= J(d\lambda\wedge dx_1 - a_{11}d\lambda\wedge dx_2 - a_{21}d\lambda\wedge dx_3 + a_{10}dx_1\wedge dx_2 + \\ &\quad + a_{20}dx_1\wedge dx_3 - (a_{11}a_{20} - a_{10}a_{21})dx_2\wedge dx_3) \\ &\quad a_{10} = u_0(x), \quad a_{11} = u_1(x) + \lambda, \\ &\quad a_{20} = v_0(x) + \lambda v_1(x), \quad a_{21} = v_2(x) + \lambda v_3(x) + \lambda^2.\end{aligned}$$

Denoting $x = x_1$, $y = x_2$, $t = x_3$, one gets the Manakov-Santini system

$$\begin{aligned}u_{xt} + u_{yy} + u_x^2 + (u - v_y)u_{xx} + v_x u_{xy} &= 0, \\ v_{xt} + v_{yy} + v_x v_{xy} + (u - v_y)v_{xx} &= 0.\end{aligned}$$

Reductions

1.The form Ω_2 is closed in the standard sense in affine gauge (J=1) - dKP. In general, the closedness in the affine gauge leads to volume-preserving vector fields.

2.Reduction $\Omega_2 \wedge d\lambda = 0$. In this case it is possible to consider Ω_1 not containing $d\lambda$. Vector fields do not contain a derivative over spectral variable. Leeds to Pavlov system and Martinez Alonso - Shabat universal hierarchy, for general Ω_m - to HCR hierarchies.

Nonlinear Riemann-Hilbert problem

Let us consider the closed Plücker form Ω_m. Being decomposable, it can be represented as

$$\Omega_m = d\Psi^0 \wedge d\Psi^1 \wedge \ldots d\Psi^{m-1} = J\tilde{\Omega}_m$$

J is some coefficient of the form in coordinates $\mathbf{x}$, $\tilde{\Omega}_m$ is a gauge-invariant (affine) factor, Ψ^k are some functions (series in λ).

Question

How to provide some simple analytic properties of the affine factor? What kind of functions Ψ^k correspond to a polynomial affine factor?

It is easy to see that $\tilde{\Omega}_m$ is invariant under diffeomorphism

$$(\Psi^0, \Psi^1, \ldots, \Psi^{m-1}) \to \mathbf{F}(\Psi^0, \Psi^1, \ldots, \Psi^{m-1})$$

Let Ψ^k be holomorphic (meromorphic) inside and outside the unit circle, having a discontinuity on it. If they satisfy a nonlinear vector Riemann-Hilbert problem (nvRHp)

$$(\Psi^0, \Psi^1, \ldots, \Psi^{m-1})_{\text{in}} = \mathbf{F}(\Psi^0, \Psi^1, \ldots, \Psi^{m-1})_{\text{out}},$$

then the affine factor $\tilde{\Omega}_m$ is holomorhic (meromorphic) in all the complex plane.

nvRHp gives a tool to construct closed Plüker forms with holomorphic (meromorphic) affine factor, generating commuting vector fields with holomorphic (meromorhic) coefficients

General hierarchy for the polynomial case

We consider the formal series of the variable λ

$$\Psi^0 = \lambda + \sum_{n=1}^{\infty} \Psi_n^0(\mathbf{t}^1, \ldots, \mathbf{t}^{m-1})\lambda^{-n}, \quad (1)$$

$$\Psi^k = \sum_{n=0}^{\infty} t_n^k (\Psi^0)^n + \sum_{n=1}^{\infty} \Psi_n^k(\mathbf{t}^1, \ldots, \mathbf{t}^{m-1})(\Psi^0)^{-n}, \quad (2)$$

where $1 \leqslant k \leqslant m-1$, depending on $m-1$ infinite sequences of independent variables $\mathbf{t}^k = (t_0^k, \ldots, t_n^k, \ldots)$, $t_0^k = x_k$, $\lambda = x_0$.
The hierarchy is generated by the relation

$$\left(J^{-1} d\Psi^0 \wedge d\Psi^1 \wedge \ldots d\Psi^{m-1}\right)_- = 0$$

where $(\cdots)_-$ denotes the projection on the part of $(\cdots)$ with negative powers in λ and $J = \pi_{0\,1\,\ldots\,m-1} = \det(\partial_l \Psi^k)_{k,l=0,\ldots,m-1}$. Generating relation represents a polynomiality condition for affine factor of the closed Plücker form, it can be provided using RH problem.

Using the Jacobian matrix

$$(\mathrm{Jac}_0) = \left(\frac{D(\Psi^0, \ldots, \Psi^{m-1})}{D(x_0, \ldots, x_{m-1})} \right), \quad \det(\mathrm{Jac}_0) = J,$$

it is possible to write Lax-Sato equation of the hierarchy inthe form

$$\partial_n^k \mathbf{\Psi} = \sum_{i=0}^{m-1} \left(((\mathrm{Jac}_0)^{-1})_{ik} (\Psi^0)^n) \right)_+ \partial_i \mathbf{\Psi}, \quad 1 \leqslant k \leqslant m-1, \tag{3}$$

where $1 \leqslant n < \infty$, $\mathbf{\Psi} = (\Psi^0, \ldots, \Psi^{m-1})$. First flows of the hierarchy read

$$\partial_1^k \mathbf{\Psi} = (\lambda \partial_k - \sum_{p=1}^{m-1} (\partial_k u_p) \partial_p - (\partial_k u_0) \partial_\lambda) \mathbf{\Psi}, \quad 1 \leqslant k \leqslant m-1, \tag{4}$$

where $u_0 = \Psi_1^0$, $u_k = \Psi_1^k$, $1 \leqslant k \leqslant m-1$.

A compatibility condition for any pair of linear equations (e.g., with ∂_1^k and ∂_1^q, $k \neq q$) implies closed nonlinear N-dimensional system of PDEs for the set of functions u_k, u_0, which can be written in the form

$$\begin{aligned} &\partial_1^k \partial_q \hat{u} - \partial_1^q \partial_k \hat{u} + [\partial_k \hat{u}, \partial_q \hat{u}] = (\partial_k u_0)\partial_q - (\partial_q u_0)\partial_k, \\ &\partial_1^k \partial_q u_0 - \partial_1^q \partial_k u_0 + (\partial_k \hat{u})\partial_q u_0 - (\partial_q \hat{u})\partial_k u_0 = 0, \end{aligned} \tag{5}$$

where $\hat{u}$ is a vector field, $\hat{u} = \sum_{p=1}^{m-1} u_p \partial_p$. For $m = 3$ this system after volume-preservation reduction corresponds to the Dunajski system (generalizing heavenly equation).

General heavenly equation

*B. Doubrov, E.V. Ferapontov, On the integrability of symplectic Monge-Ampère equations, Journal of Geometry and Physics, **60**(10), 1604—1616 (2010)*

$$\alpha u_{12} u_{34} + \beta u_{13} u_{24} + \gamma u_{14} u_{23} = 0,$$

where $\alpha + \beta + \gamma = 0$, subscripts denote partial derivatives.
Lax pair: (vector fields X_1, X_2 in involution)

$$X_1 = u_{34}\partial_1 - u_{13}\partial_4 + \gamma\lambda(u_{34}\partial_1 - u_{14}\partial_3),$$
$$X_2 = u_{23}\partial_4 - u_{34}\partial_2 + \beta\lambda(u_{34}\partial_2 - u_{24}\partial_3).$$

Let us consider 2-form depending on the spectral parameter

$$\Omega = \sum \omega_{ij} dx_i \wedge dx_j,$$

where $1 \leq i, j \leq 4$,

$$\omega_{ij}(\lambda, \mathbf{x}) = \left(\frac{1}{\lambda - \lambda_i} - \frac{1}{\lambda - \lambda_j} \right) w_{ij}(\mathbf{x}),$$

$w_{ij}(\mathbf{x})$ is symmetric (here subscripts don't suggest differentiation). Pl ucker conditions for 2-forms are equivalent to the relation

$$\Omega \wedge \Omega = 0,$$

which in our case gives one equation

$$\omega_{23}\omega_{14} - \omega_{13}\omega_{24} + \omega_{12}\omega_{34} = 0,$$

and for $w_{ij}(\mathbf{x})$ we have

$$(\lambda_3 - \lambda_2)(\lambda_4 - \lambda_1) w_{23} w_{14} - (\lambda_3 - \lambda_1)(\lambda_4 - \lambda_2) w_{13} w_{24} \\ + (\lambda_2 - \lambda_1)(\lambda_4 - \lambda_3) w_{12} w_{34} = 0$$

(no spectral parameter!)

Closedness conditions

Let us also suggest that Ω is closed, $\omega_{[ij,k]}=0$. For $w_{ij}(\mathbf{x})$ we have

$$\frac{\partial_1 w_{23}-\partial_3 w_{12}}{\lambda-\lambda_2}+\frac{\partial_2 w_{13}-\partial_1 w_{23}}{\lambda-\lambda_3}+\frac{\partial_3 w_{12}-\partial_2 w_{13}}{\lambda-\lambda_1}=0,$$
$$\frac{\partial_1 w_{24}-\partial_3 w_{12}}{\lambda-\lambda_2}+\frac{\partial_2 w_{14}-\partial_1 w_{24}}{\lambda-\lambda_4}+\frac{\partial_3 w_{12}-\partial_2 w_{14}}{\lambda-\lambda_1}=0,$$
$$\frac{\partial_1 w_{34}-\partial_3 w_{14}}{\lambda-\lambda_4}+\frac{\partial_2 w_{13}-\partial_1 w_{34}}{\lambda-\lambda_3}+\frac{\partial_3 w_{14}-\partial_2 w_{13}}{\lambda-\lambda_1}=0,$$
$$\frac{\partial_1 w_{23}-\partial_3 w_{24}}{\lambda-\lambda_2}+\frac{\partial_2 w_{34}-\partial_1 w_{23}}{\lambda-\lambda_3}+\frac{\partial_3 w_{24}-\partial_2 w_{34}}{\lambda-\lambda_4}=0.$$

These equations imply the existence of the potential

$$\Theta : w_{ij}=\Theta_{,ij}$$

and for arbitrary potential Θ $w_{ij}=\Theta_{,ij}$ satisfy the closedness equations. The constants in w_{ij} do not affect the closedness, correspond to the term in Θ quadratic in x_i $(\sum c_{ij}x_i x_j)$.

Proposition

Let us consider 2-form depending on the spectral parameter

$$\Omega = \sum \left(\frac{1}{\lambda - \lambda_i} - \frac{1}{\lambda - \lambda_j} \right) w_{ij}(\mathbf{x}) dx_i \wedge dx_j,$$

where $1 \leqslant i, j \leqslant 4$, $w_{ij}(\mathbf{x})$ *is symmetric. The conditions*

$$\Omega \wedge \Omega = 0,$$
$$d\Omega = 0$$

are equivalent to the equation ($w_{ij} = \Theta_{,ij}$)

$$(\lambda_3 - \lambda_2)(\lambda_4 - \lambda_1)\Theta_{,23}\Theta_{,14} - (\lambda_3 - \lambda_1)(\lambda_4 - \lambda_2)\Theta_{,13}\Theta_{,24} + (\lambda_2 - \lambda_1)(\lambda_4 - \lambda_3)\Theta_{,12}\Theta_{,34} = 0.$$

A limit to the first heavenly equation

General heavenly equation plays a role of generating equation for the heavenly equation hierarchy (Takasaki).
Let us consider a limit

$$\lambda_1, \lambda_2 \to a, \quad \lambda_3, \lambda_4 \to b$$

for the potential Θ of the form

$$\Theta = c_{12} \frac{x_1 x_2}{\lambda_1 - \lambda_2} + c_{34} \frac{x_3 x_4}{\lambda_3 - \lambda_4} + \tilde{\Theta}$$

General heavenly equation in this limit gives Plebański first heavenly equation in the form

$$\tilde{\Theta}_{,13} \tilde{\Theta}_{,24} - \tilde{\Theta}_{,23} \tilde{\Theta}_{,14} = \frac{c_{12} c_{34}}{(b-a)^2}.$$

Corresponding 2-form Ω is also obtained in this limit.

The Lax pair: vector fields in involution

General vector fields correspond to projectively closed Ω, for Ω closed in the standard sense the basic fields can be chosen divergence-free

$$U_{ijk} = \left(\frac{1}{\lambda-\lambda_i} - \frac{1}{\lambda-\lambda_j}\right) w_{ij}\partial_k + \left(\frac{1}{\lambda-\lambda_j} - \frac{1}{\lambda-\lambda_k}\right) w_{jk}\partial_i + \left(\frac{1}{\lambda-\lambda_k} - \frac{1}{\lambda-\lambda_i}\right) w_{ki}\partial_j$$

Linear span is 2-dimensional (due to Plüker relation), arbitrary pair of independent U_{ijk} can be taken as a Lax pair. Divergence-free condition implies the existence of potential $\Theta : w_{ij} = \Theta_{,ij}$.

In equivalent form, taking $U_{ijk} \to (\lambda - \lambda_i)(\lambda - \lambda_j)(l - \lambda_k)U_{ijk}$, we get polynomial fields of the first order in spectral parameter:

$$U_{ijk} = (\lambda_i - \lambda_j)w_{ij}(\lambda - \lambda_k)\partial_k + (\lambda_j - \lambda_k)w_{jk}(\lambda - \lambda_i)\partial_i + (\lambda_k - \lambda_i)w_{ki}(\lambda - \lambda_k)\partial_j.$$

The divergence-free condition gives $w_{ij} = \Theta_{,ij}$, and after Möbius transformation it is possible to get the Lax pair in Doubrov-Ferapontov form.

Dual space of one-forms

Due to Plüker relations, Ω can be represented as

$$\Omega = \psi \wedge \phi.$$

Space of 1-forms (Grassmannian point in cotangent space)

$$\phi_p = \left(\frac{1}{\lambda - \lambda_i} - \frac{1}{\lambda - \lambda_p}\right) w_{pi} dx_i + \left(\frac{1}{\lambda - \lambda_j} - \frac{1}{\lambda - \lambda_p}\right) w_{pj} dx_j + \left(\frac{1}{\lambda - \lambda_k} - \frac{1}{\lambda - \lambda_p}\right) w_{pk} dx_k$$

It is possible to the basis of polynomial forms of the first order in λ, e.g.

$$\phi^1_{34} = (\lambda - \lambda_2)(\lambda - \lambda_3)(\lambda - \lambda_4)(w_{14}\phi_3 - w_{13}\phi_4)$$

Gindikin construction

*S.G. Gindikin, Some solutions of the selfdual Einstein equations, Func. Anal. and Appl. (in Russian),**19**(3), 58–60 (1985)*

citation:

Construction of complex solutions of self-dual Einstein equations is equivalent to construction of quadratic (in t) bundles of holomorphic two-forms $F(t) = t^2F_2 + tF_1 + F_0$, $t \in \mathbb{C}$, on the four-dimensional complex manifold M, satisfying the conditions

$$\begin{aligned} F(t) \wedge F(t) &= 0 \quad \text{for all } t; \\ dF(t) &= 0; \\ F(t) \wedge F(s) &\neq 0 \quad \text{for } t \neq s. \end{aligned}$$

... Due to the first condition $F(t)$ can be represented as

$$F(t) = (\phi_0 + t\phi_1) \wedge (\psi_0 + t\psi_1),$$

where ϕ_i, ψ_i are 1-forms. Then the third condition guarantees non-degeneracy of the metrics

$$g = \phi_0\psi_1 - \phi_1\psi_0,$$

and the second condition implies that it satisfies self-dual Einstein equations.

The form

$$F(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)(\lambda - \lambda_4)\Omega$$

is quadratic in λ and satisfies the required properties, the metrics can be constructed explicitly.

$\bar{\partial}$-dressing scheme

Ω can be represented in a form

$$\Omega = dS^1 \wedge dS^2.$$

Two properties of Ω are now identically satisfied (it is Plüker and closed), now the problem is to construct S^1, S^2 to get Ω with the necessary analytic properties.

L.V. Bogdanov, B.G. Konopelchenko, On the $\bar{\partial}$-dressing method applicable to heavenly equation, Physics Letters A, Volume 345, Issues 1–3, 26 September 2005, Pages 137–143

Nonlinear vector $\bar{\partial}$ problem in some region G,

$$\begin{aligned} \bar{\partial} S^1 &= W_{,2}(z, \bar{z}; S^1, S^2), \\ \bar{\partial} S^2 &= -W_{,1}(z, \bar{z}; S^1, S^2) \end{aligned}$$

This problem provides analyticity of the form $\Omega = dS^1 \wedge dS^2$ in G.
We search for solutions of the form

$$S^i = S_0^i + \tilde{S}^i,$$

where $\tilde{S}^i$ are regular in $\mathbb{C}$, analytic outside G and decrease at infinity, S_0^i are analytic in G (normalization),

$$S_0^1 = \sum \frac{c_k^1 x_k}{\lambda - \lambda_k}, \quad S_0^2 = \sum \frac{c_k^2 x_k}{\lambda - \lambda_k}.$$

Then the form Ω has the required analytic structure.

Formula for Θ

The $\bar\partial$ problem can be obtained by variation of the action

$$f = \frac{1}{2\pi \mathrm{i}} \iint_G \left(\widetilde{S}^2 \bar\partial \widetilde{S}^1 - W(z, \bar z, S^1, S^2) \right) dz \wedge d\bar z, \tag{6}$$

where one should consider independent variations of $\widetilde{\mathbf{S}}$, possessing required analytic properties, keeping $\mathbf{S}_0$ fixed.

Theorem

The function

$$\Theta(\mathbf{x}) = \frac{1}{2\pi i} \iint_{\mathbb{C}} (\widetilde{S}^2(\mathbf{x}) \bar\partial \widetilde{S}^1(\mathbf{x}) + S_0^2(\mathbf{x}) \bar\partial S_0^1(\mathbf{x}) \\ - W(z, \bar z, S^1(\mathbf{x}), S^2(\mathbf{x}))) dz \wedge d\bar z,$$

i.e., the action evaluated on the solution of the $\bar\partial$ problem plus a term quadratic in x_i, is a solution of the Doubrov-Ferapontov general heavenly equation.

Multidimesional hyper-Kähler case

The structure of Ω the same,

$$\begin{aligned} \Omega \wedge \cdots \wedge \Omega &= 0 \quad (N \text{ times}), \\ d\Omega &= 0. \end{aligned}$$

Closedness implies the existence of Θ.
$2N$-dimensional homogeneous of degree N "general hyper-Kähler equation"

$$\sum \epsilon_{i_1 \dots i_{2N}} (\lambda_{i_1} - \lambda_{i_2}) \times \cdots \times (\lambda_{i_{2N-1}} - \lambda_{i_{2N}}) \Theta_{,i_1 i_2} \times \cdots \times \Theta_{,i_{2N-1} i_{2N}} = 0,$$

$1 \leqslant i_k \leqslant 2N$. Generating equation for multidimensional hyper-Kähler hierarchy (Takasaki).
Ω can be represented as

$$\Omega = S^1 \wedge S^2 + \cdots + S^{2N-3} \wedge S^{2N-2}$$

Similar to the heavenly case: $\bar{\partial}$-dressing, formula for Θ.

THANK YOU!