# Waves' Instabilities on a Discrete Grid.

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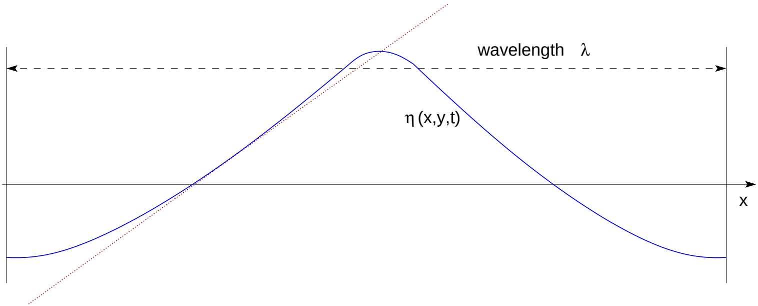
Chernogolovka, Russia

# Water waves. Problem formulation.

Let us consider a potential flow of an ideal fluid of infinite depth with a free surface. We use standard notations for velocity potential $\phi(\vec{r}, z, t), \vec{r} = (x, y); \vec{v} = \nabla\phi$ and surface elevation $\eta(\vec{r}, t)$.




Steepness of the surface $\mu = \sqrt{\langle|\nabla\eta(\vec{r}, t)|^2\rangle} \simeq 0.1$ — average slope of the surface.

# Energy of the system

Fluid flow is incompressible $(\nabla \vec{v}) = \Delta\phi = 0$. The total energy of the system can be presented in the following form

$$H = T + U,$$

Kinetic energy:

$$T = \frac{1}{2}\int \mathrm{d}^2 r \int\limits_{-\infty}^{\eta} (\nabla\phi)^2 \mathrm{d}z,$$

Potential energy due to gravity:

$$U = \frac{1}{2} g \int \eta^2 \mathrm{d}^2 r,$$

here $g$ is the gravity acceleration.

## Hamiltonian expansion.

It was shown by Zakharov (1966) that under these assumptions the fluid is a Hamiltonian system

$$\frac{\partial \eta}{\partial t} = \frac{\delta H}{\delta \psi}, \quad \frac{\partial \psi}{\partial t} = -\frac{\delta H}{\delta \eta},$$

where $\psi = \phi(\vec{r}, \eta(\vec{r}, t), t)$ is a velocity potential on the surface of the fluid. In order to calculate the value of $\psi$ we have to solve the Laplace equation in the domain with varying surface $\eta$. One can simplify the situation, using the expansion of the Hamiltonian in powers of "steepness" (here $\Delta = \nabla^2$ and $\hat{k} = \sqrt{-\Delta}$)

$$\begin{aligned} H = & \frac{1}{2}\int \left(g\eta^2 + \psi\hat{k}\psi\right) \mathrm{d}^2 r + \\ & + \frac{1}{2}\int \eta\left[|\nabla\psi|^2 - (\hat{k}\psi)^2\right] \mathrm{d}^2 r + \\ & + \frac{1}{2}\int \eta(\hat{k}\psi)\left[\hat{k}(\eta(\hat{k}\psi)) + \eta\Delta\psi\right] \mathrm{d}^2 r. \end{aligned}$$

# Dynamical equations.

In this case dynamical equations acquire the following form

$$
\begin{aligned}
\dot{\eta} &= \hat{k}\psi - (\nabla(\eta\nabla\psi)) - \hat{k}[\eta\hat{k}\psi] + \\
&+ \hat{k}(\eta\hat{k}[\eta\hat{k}\psi]) + \frac{1}{2}\Delta[\eta^2\hat{k}\psi] + \frac{1}{2}\hat{k}[\eta^2\Delta\psi] - D_{\vec{r}}, \\
\dot{\psi} &= -g\eta - \frac{1}{2}\left[(\nabla\psi)^2 - (\hat{k}\psi)^2\right] - \\
&- [\hat{k}\psi]\hat{k}[\eta\hat{k}\psi] - [\eta\hat{k}\psi]\Delta\psi - D_{\vec{r}} + F_{\vec{r}}.
\end{aligned}
$$

Here $D_{\vec{r}}$ is some artificial damping term used to provide dissipation at small scales; $F_{\vec{r}}$ is a pumping term corresponding to external force (having in mind wind blow, for example). Let us introduce Fourier transform

$$
\psi_{\vec{k}} = \frac{1}{2\pi}\int \psi_{\vec{r}} \mathrm{e}^{\mathrm{i}\vec{k}\vec{r}} \mathrm{d}^2 r, \quad \eta_{\vec{k}} = \frac{1}{2\pi}\int \eta_{\vec{r}} \mathrm{e}^{\mathrm{i}\vec{k}\vec{r}} \mathrm{d}^2 r.
$$

# Canonical variables.

$\psi(\vec{r},t)$ and $\eta(\vec{r},t)$ are real valued functions, $\Rightarrow \psi_{\vec{k}} = \psi^*_{-\vec{k}}, \eta_{\vec{k}} = \eta^*_{-\vec{k}}$ — Hermitian symmetry.

It is convenient to introduce the canonical (normal) variables $a_{\vec{k}}$ as shown below

$$a_{\vec{k}} = \sqrt{\frac{\omega_k}{2k}}\eta_{\vec{k}} + \mathrm{i}\sqrt{\frac{k}{2\omega_k}}\psi_{\vec{k}}, \text{ where } \omega_k = \sqrt{gk}.$$

$$\dot{a}_{\vec{k}} = -\mathrm{i}\frac{\delta H}{\delta a^*_{\vec{k}}} \text{ — Hamiltonian equations,}$$

$a_{\vec{k}}$ — is an elementary excitation (plane wave).

# Resonant conditions

Let us get rid of the linear part:

$$(a_{\vec{k}_1} a_{\vec{k}_2} a^*_{\vec{k}_0} + a^*_{\vec{k}_1} a^*_{\vec{k}_2} a_{\vec{k}_0})\delta(\vec{k}_1 + \vec{k}_2 - \vec{k}_0)$$

$$a_{\vec{k}}(t) = A_{\vec{k}}(t)\mathrm{e}^{\mathrm{i}\omega_k t} \Rightarrow a^*_{\vec{k}_0} a_{\vec{k}_1} a_{\vec{k}_2} = A^*_{\vec{k}_0} A_{\vec{k}_1} A_{\vec{k}_2} \mathrm{e}^{\mathrm{i}(\omega_{k_0} - \omega_{k_1} - \omega_{k_2})t}$$

Resonant conditions for 3-waves interaction (decaying and merging):

$$\omega_{k_0} = \omega_{k_1} + \omega_{k_2}, \quad \vec{k}_0 = \vec{k}_1 + \vec{k}_2.$$

Resonant conditions for 4-waves interaction (two into two scattering):

$$\omega_{k_1} + \omega_{k_2} = \omega_{k_3} + \omega_{k_4}, \quad \vec{k}_1 + \vec{k}_2 = \vec{k}_3 + \vec{k}_4. \qquad (1)$$

# 3-waves interaction. Capillary waves.

In the case of capillary waves on the surface of deep fluid the dispersion is given by

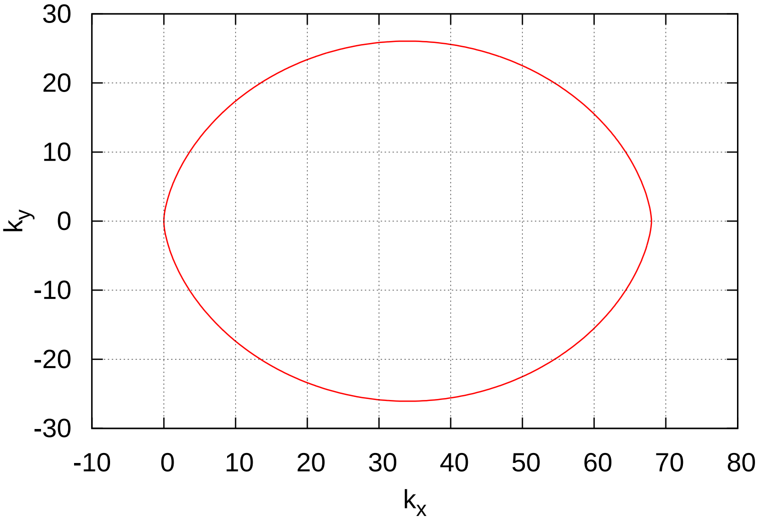
$$\omega_k = \sqrt{\sigma k^3}, \tag{2}$$

here $\sigma$ is the surface tension coefficient.
Let us consider three-waves process, corresponding to decay and merging of waves. One can get resonance condition

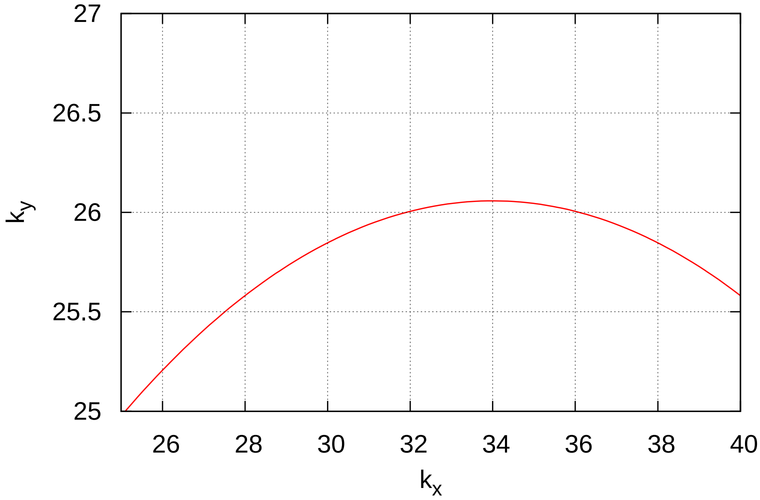
$$\omega_{k_1} + \omega_{k_2} = \omega_{k_0}, \quad \vec{k}_1 + \vec{k}_2 = \vec{k}_0. \tag{3}$$

To be more specific let us consider decay of wave $A_{\vec{k}_0}$ on two waves $A_{\vec{k}_1}$ and $A_{\vec{k}_2}$.

# Resonant manifold for decay of initial capillary wave with $k_0 = 68$.



# Resonant manifold for decay of initial capillary wave with $k_0 = 68$. Zoom.



# Governing system of equations.

Dynamical equations lead to the system of ordinary differential equations

$$
\begin{aligned}
\dot{A}_{\vec{k}_0} &= -\frac{i}{2}\frac{2\pi}{L_x L_y} C^{\vec{k}_0}_{\vec{k}_1 \vec{k}_2} A_{\vec{k}_1} A_{\vec{k}_2} e^{i\Omega^{k_0}_{k_1 k_2} t},\\
\dot{A}_{\vec{k}_1} &= -i\frac{2\pi}{L_x L_y} C^{\vec{k}_0}_{\vec{k}_1 \vec{k}_2} A^*_{\vec{k}_2} A_{\vec{k}_0} e^{-i\Omega^{k_0}_{k_1 k_2} t},\\
\dot{A}_{\vec{k}_2} &= -i\frac{2\pi}{L_x L_y} C^{\vec{k}_0}_{\vec{k}_1 \vec{k}_2} A^*_{\vec{k}_1} A_{\vec{k}_0} e^{-i\Omega^{k_0}_{k_1 k_2} t},
\end{aligned}
\tag{4}
$$

Here $\Omega^{k_0}_{k_1 k_2} = \omega_{k_1} + \omega_{k_2} - \omega_{k_0}$ is a mismatch of frequencies.

# Growth rate and conditions.

Let us suppose $A_{\vec{k}_1}, A_{\vec{k}_2}$ are small with respect to decaying wave $\left|A_{\vec{k}_0}\right| \gg \max(|A_{\vec{k}_1}|, |A_{\vec{k}_2}|)$ at the initial moment of time $t = 0$. Then in the beginning of harmonics growth equations (4) can be linearized. In assumption ($A_{\vec{k}_0} \simeq const$) we have growing (if several conditions are fulfilled) solution

$$A_{\vec{k}_{1,2}}(t) = A_{\vec{k}_{1,2}}(0)e^{\lambda t}, \qquad (5)$$

here

$$\lambda = -\frac{i}{2}\Omega^{k_0}_{k_1 k_2} + \sqrt{\left|\frac{2\pi}{L_x L_y} C^{\vec{k}_0}_{\vec{k}_1 \vec{k}_2} A_{\vec{k}_0}\right|^2 - \left(\frac{1}{2}\Omega^{k_0}_{k_1 k_2}\right)^2}. \qquad (6)$$

One can see, that if condition

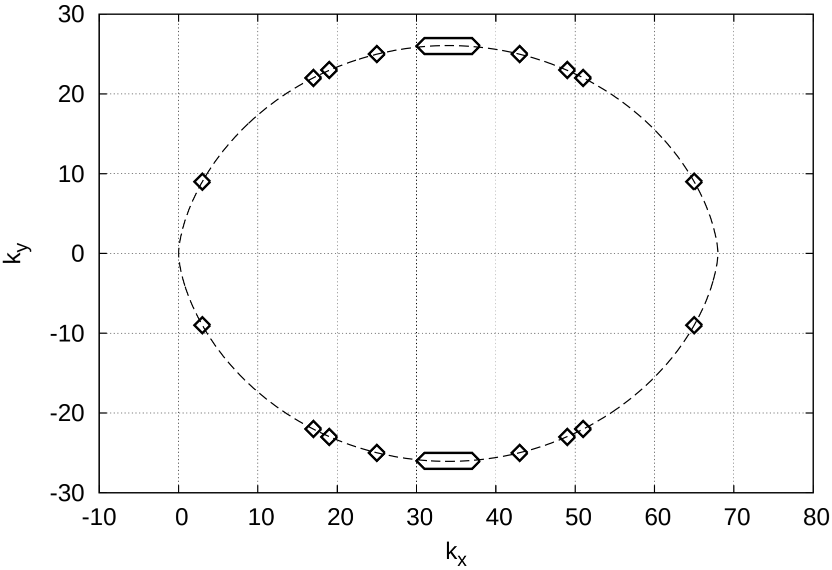
$$\left|\frac{2\pi}{L_x L_y} C^{\vec{k}_0}_{\vec{k}_1 \vec{k}_2} A_{\vec{k}_0}\right| > \left|\frac{1}{2}\Omega^{k_0}_{k_1 k_2}\right| \qquad (7)$$

is fulfilled, in the vicinity of resonant curve harmonics grow exponentially.

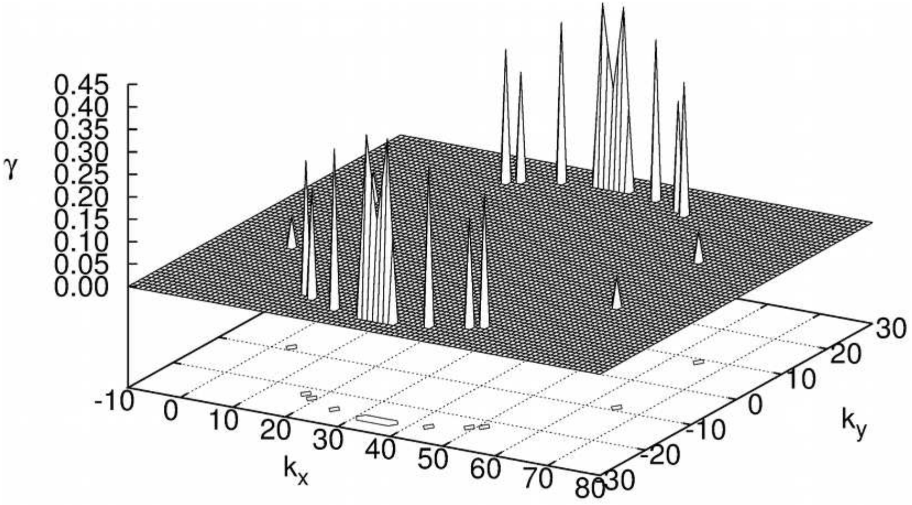
# Simulation setup.

System of dynamical equations was simulated in the domain $L_x = L_y = 2\pi$. Surface tension coefficient $\sigma = 1$. Number of grid points was $512 \times 512$. A monochromatic wave of amplitude $|a_{\vec{k}_0}| = 4 \times 10^{-3}$ was taken as initial conditions. Its wave number vector $\vec{k}_0 = (0; 68)$. All other harmonics were of amplitude $|a_{\vec{k}}| \sim 10^{-12}$ and with random phase.

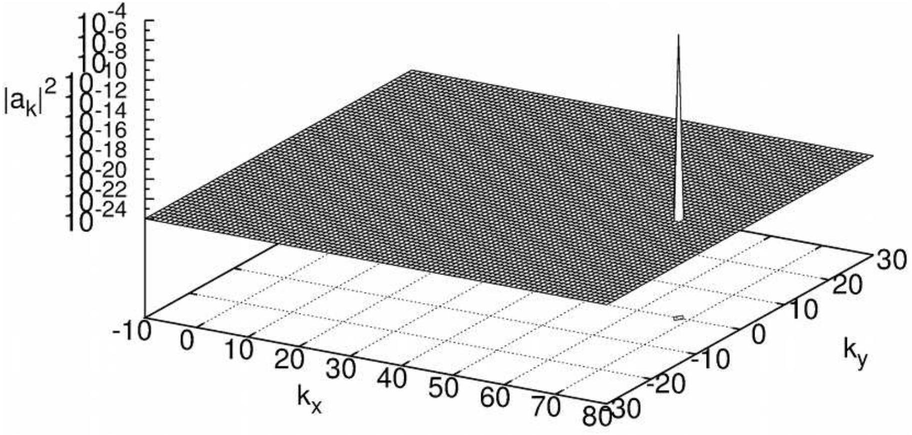
# Cappilary waves. Matrix element.



# Cappilary waves. Matrix element.



# $k_0 = 68.\ T = 0.$



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10^-6
10^-8
10^-10
10^-12
10^-14
10^-16
10^-18
10^-20
10^-22
10^-24
-10
0
10
20
30
40
50
60
70
80
$k_x$
-30
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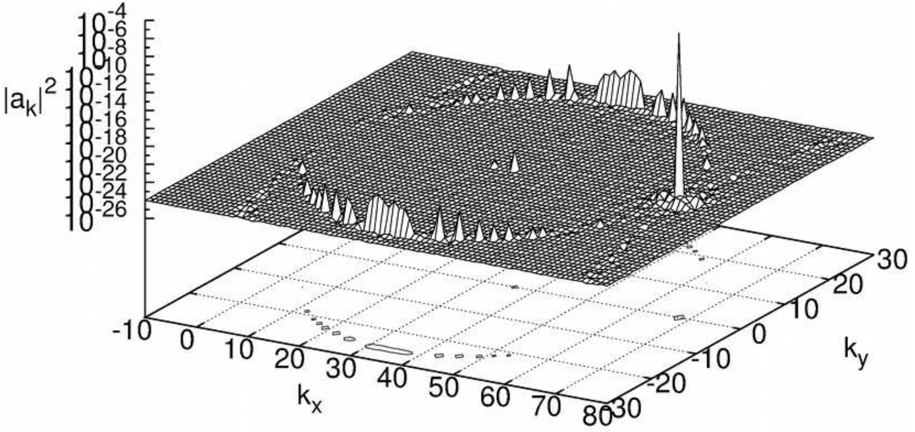
0
10
20
30
$k_y$

# $k_0 = 68.$ $T = 318\,T_0.$



$|a_k|^2$
10-4
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10-12
10-14
10-16
10-18
10-20
10-22
10-24
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10
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50
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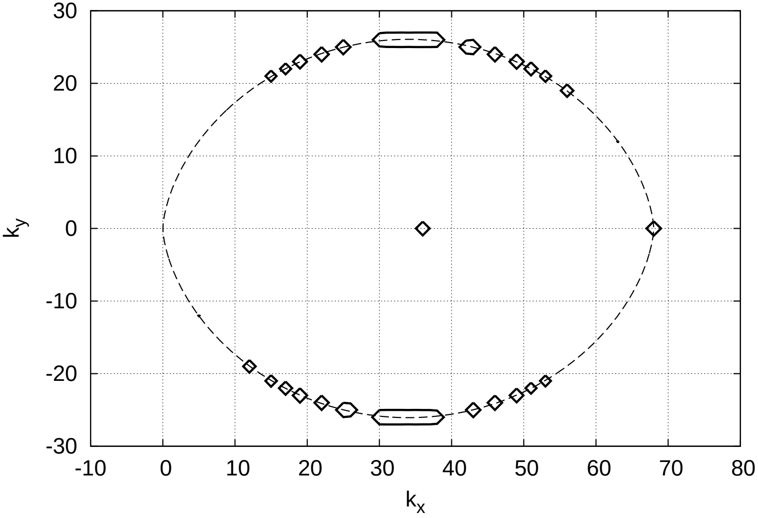
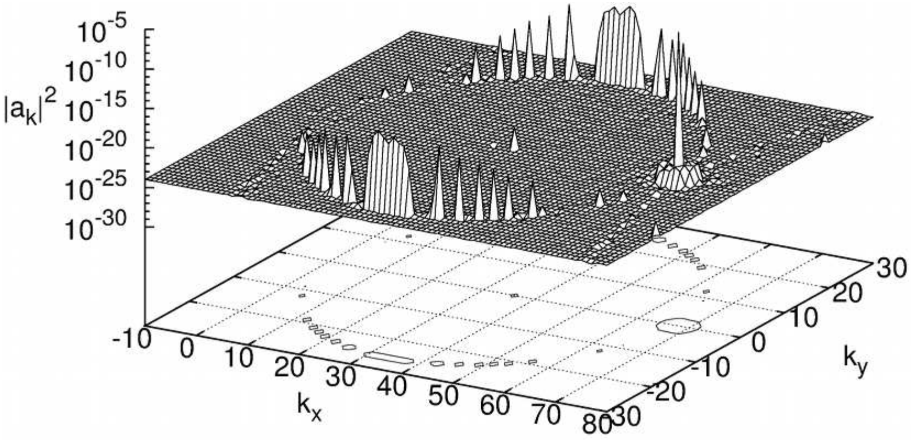
20
30
$k_y$

## $k_0 = 68.\;\; T = 318\,T_0.$



# $k_0 = 68.\ \ T = 794\, T_0.$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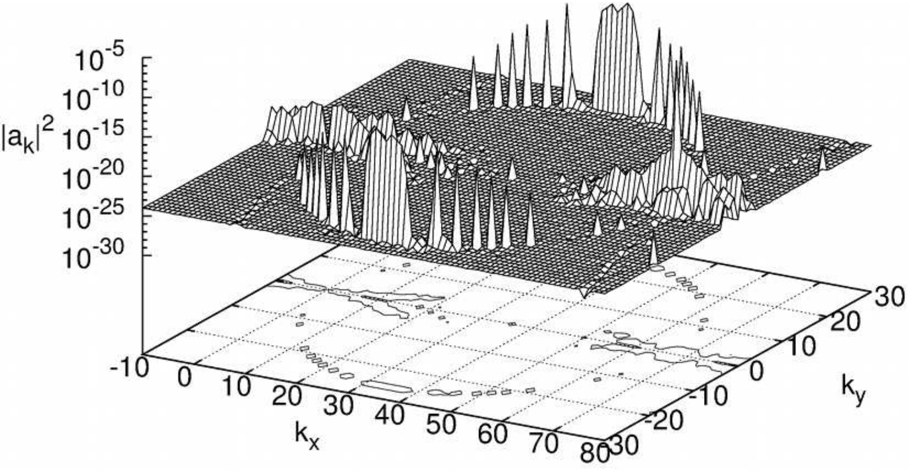
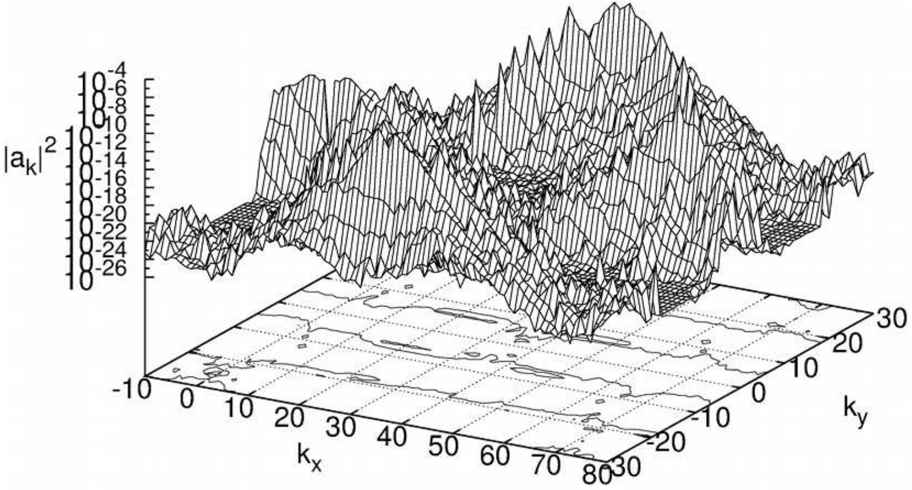
$k_y$

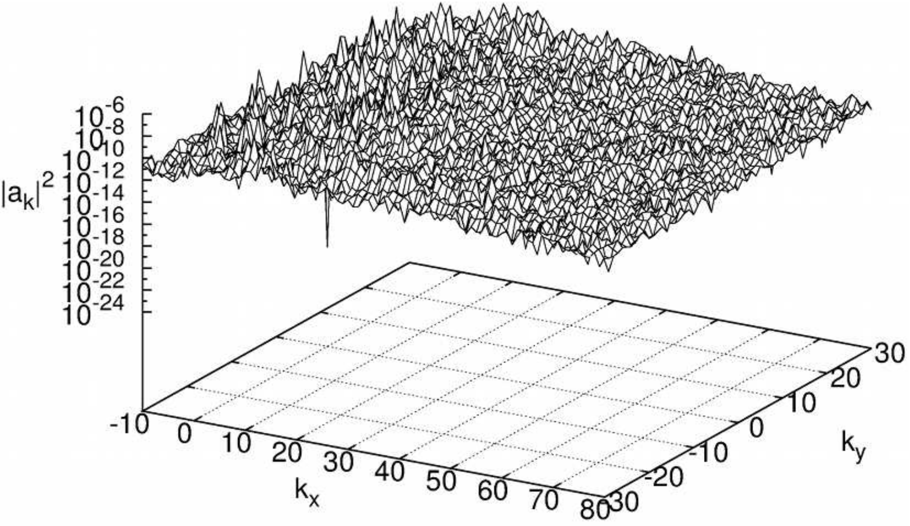
# $k_0 = 68.\ \ T = 1112\, T_0.$



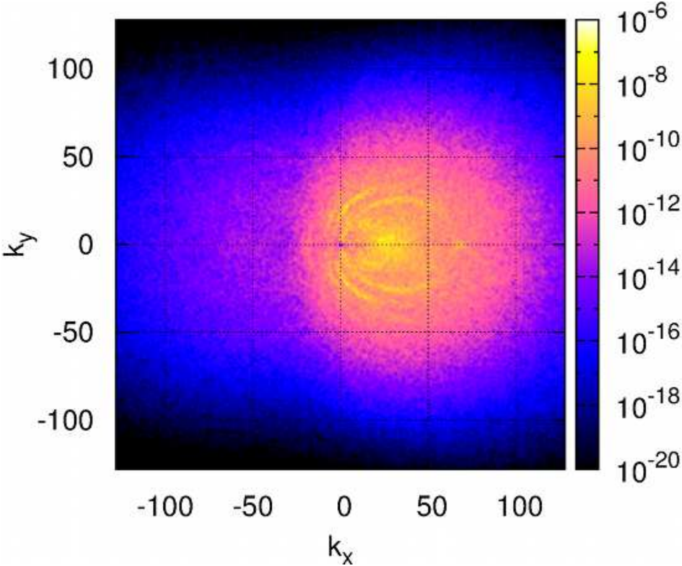
# $k_0 = 68.\ \ T = 1589\,T_0.$



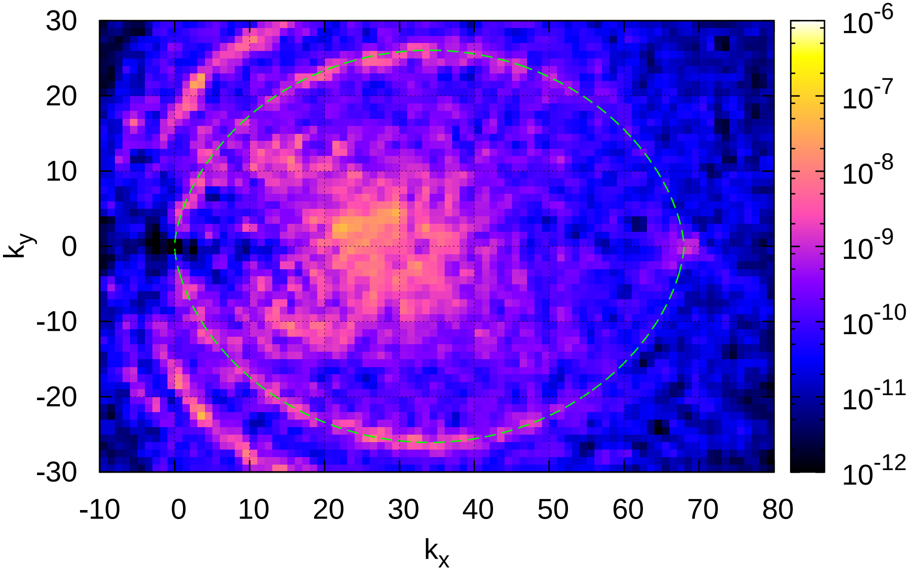
# $k_0 = 68.\;\; T = 144488\, T_0.$



## $k_0 = 68.\ \ T = 144488\, T_0.$



# $k_0 = 68.$ $T = 144488\,T_0.$



# 4-waves interaction. Gravity waves.

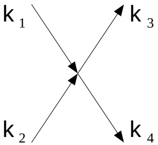
In the case of gravity waves on the surface of deep fluid the dispersion is the following

$$\omega_k = \sqrt{gk}, \tag{8}$$

here $g$ is a gravity acceleration. Here and further let us suppose $g = 1$.
In this case dispersion is of nondecay type have no real nontrivial solutions, and main process is four-wave scattering. Therefore one can make a substitution to eliminate third order terms corresponding to the decay process. This is the reason why we have to use Hamiltonian expansion up to forth order in the case of gravity waves.
Let us consider the same initial conditions as in the case of monochromatic capillary wave decay, i.e. one monochromatic wave and random phase noise of small amplitude.

# Gravity waves. Resonant condition.

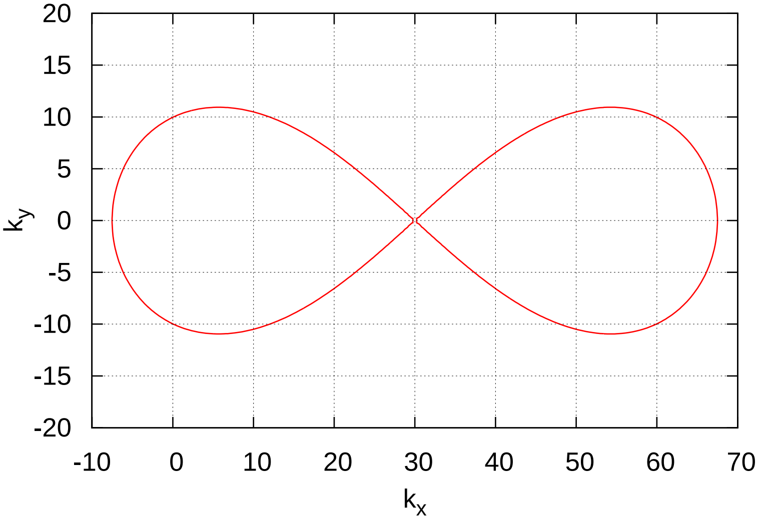
The main processes involve large amplitude of initial wave most times. In this case one wave to three and invert processes are much weaker than scattering two waves with the same amplitude and the same wave vector to two other waves.



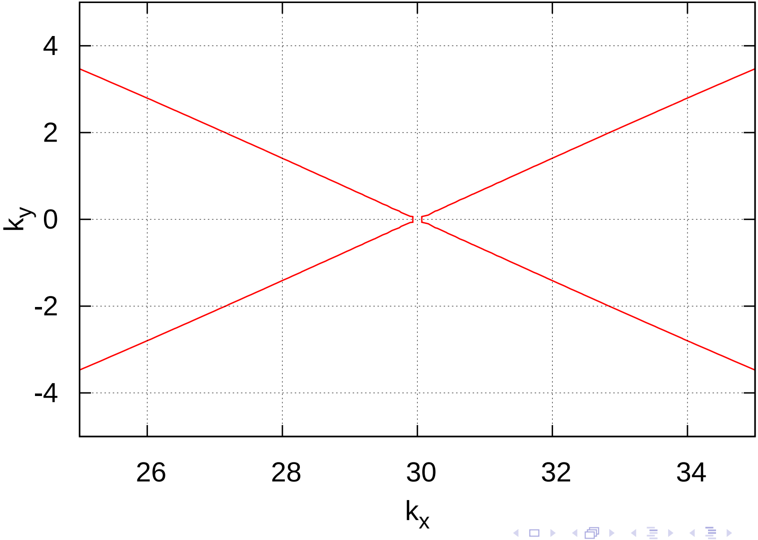

Resonance conditions for such process $(\vec{k}_3 = \vec{k}_4 = \vec{k}_0)$

$$\omega_{k_1} + \omega_{k_2} = 2\omega_{k_0}, \quad \vec{k}_1 + \vec{k}_2 = 2\vec{k}_0. \tag{9}$$

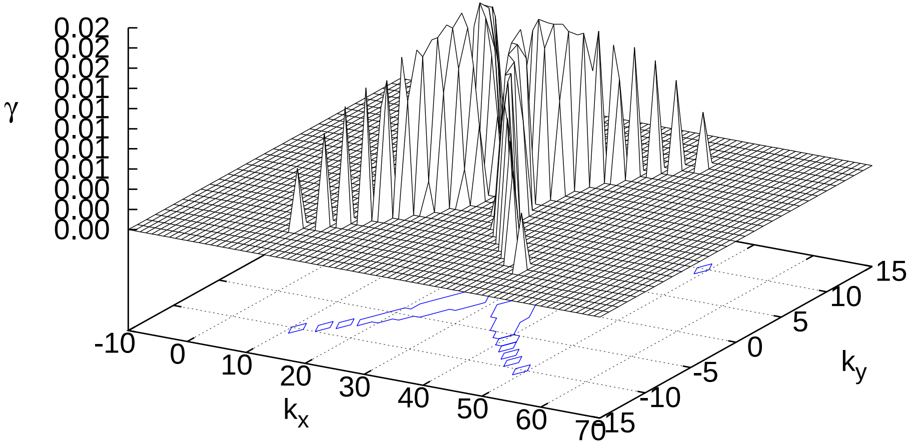
# Resonant manifold for scattering of initial gravity wave with $k_0 = 30$ (Phillips curve).



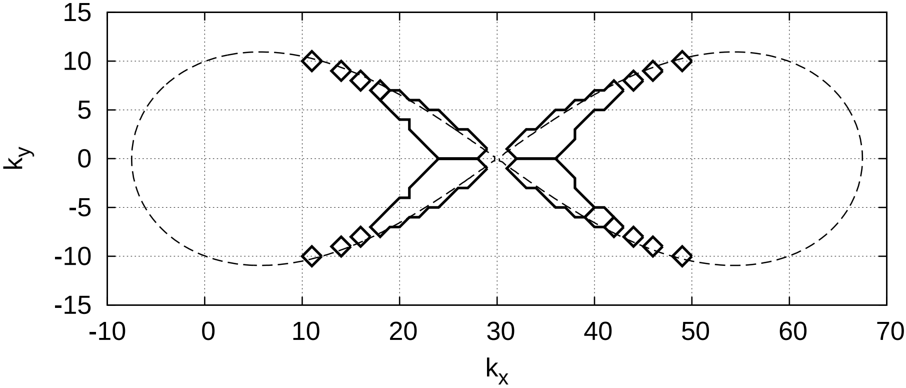
# Resonant manifold for scattering of initial gravity wave with $k_0 = 30$. Zoom.



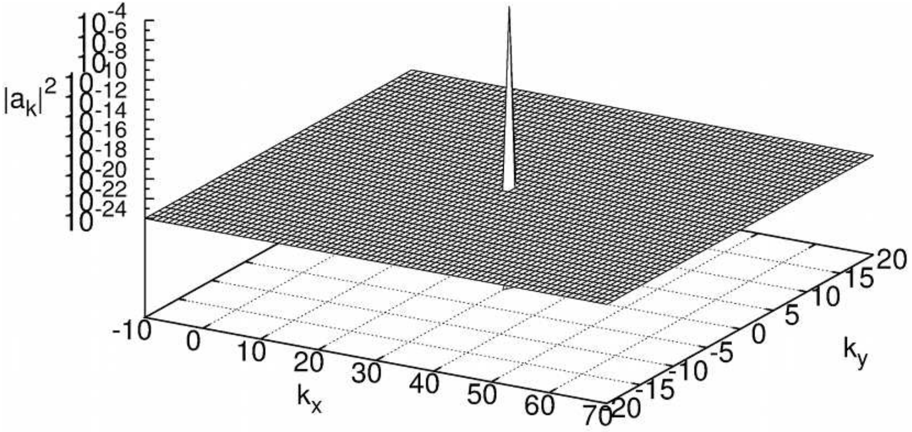
# Gravity waves. Matrix element.



# Gravity waves. Matrix element.



# $k_0 = 30.\ T = 0.$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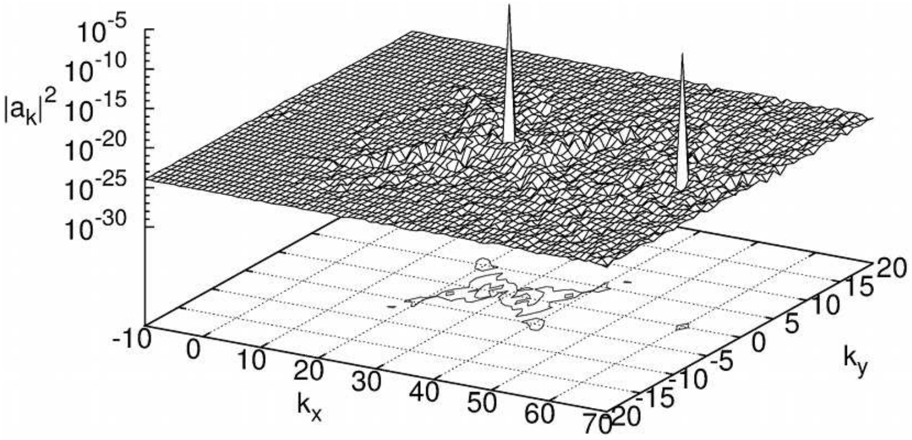
$k_y$

# $k_0 = 30.\ T = 43T_0.$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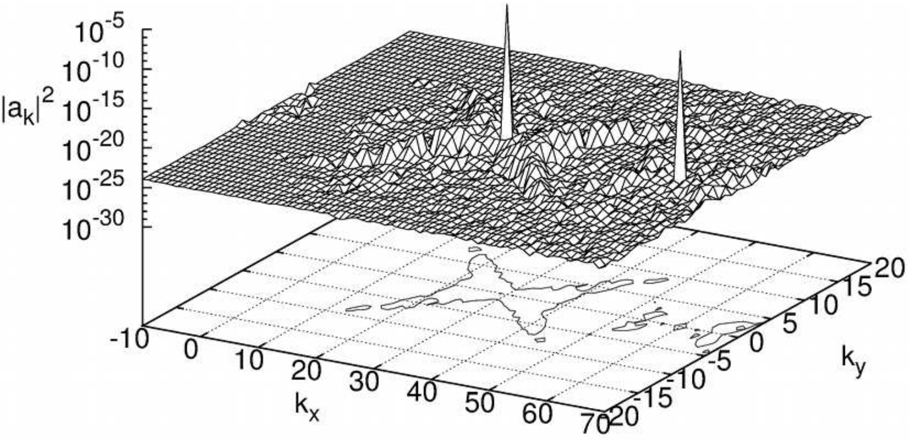
$k_y$

# $k_0 = 30.\ \ T = 87T_0.$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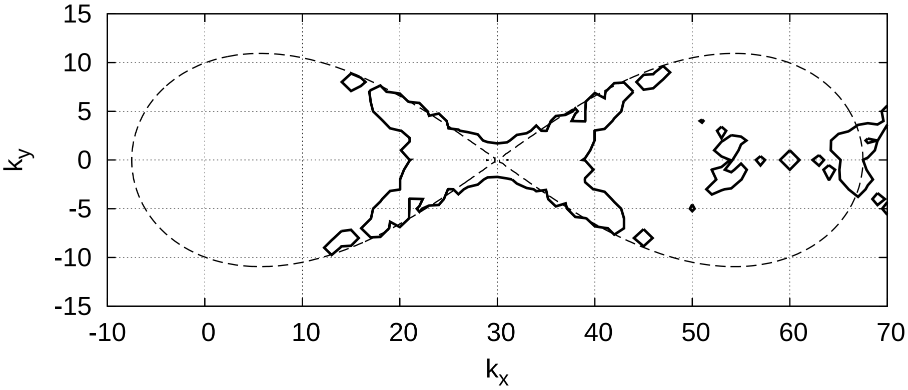
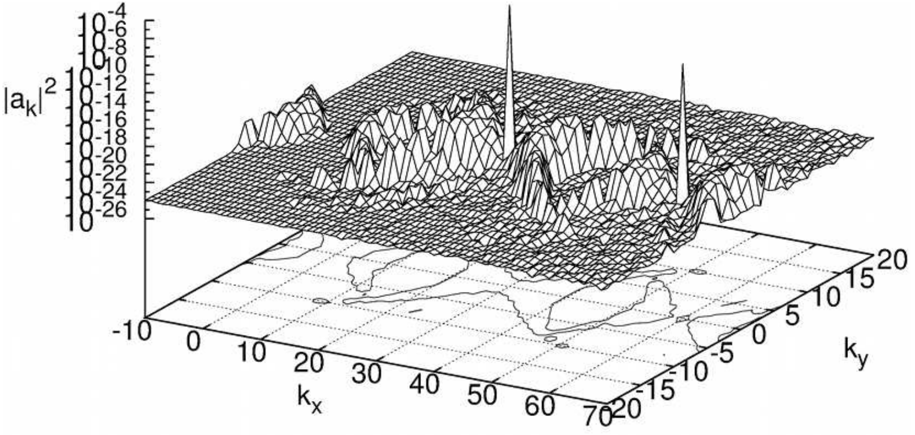
$k_y$

## $k_0 = 30.\ \ T = 87\,T_0.$



## $k_0 = 30.\ \ T = 174\,T_0.$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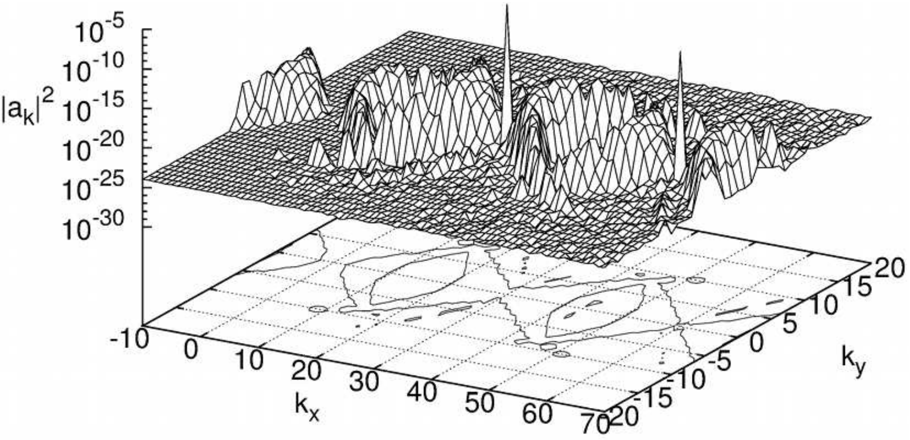
10^-4 10^-6 10^-8 10^-10 10^-12 10^-14 10^-16 10^-18 10^-20 10^-22 10^-24 10^-26 10
-10 0 10 20 30 40 50 60 70
$k_x$
-20 -15 -10 -5 0 5 10 15 20
$k_y$

## $k_0 = 30.\ T = 261\,T_0.$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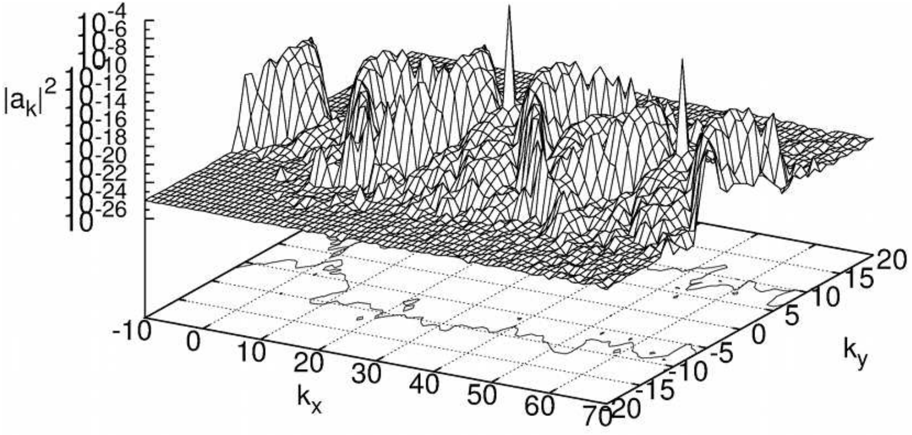
$k_y$

# $k_0 = 30.\ \ T = 348\,T_0.$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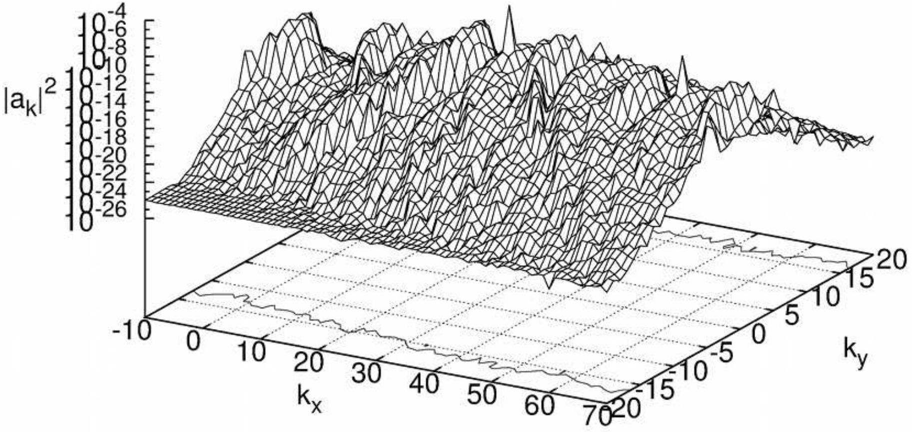
-10 0 10 20 30 40 50 60 70
$k_x$
-20 -15 -10 -5 0 5 10 15 20
$k_y$

# $k_0 = 30.\ \ T = 435\, T_0.$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$k_x$
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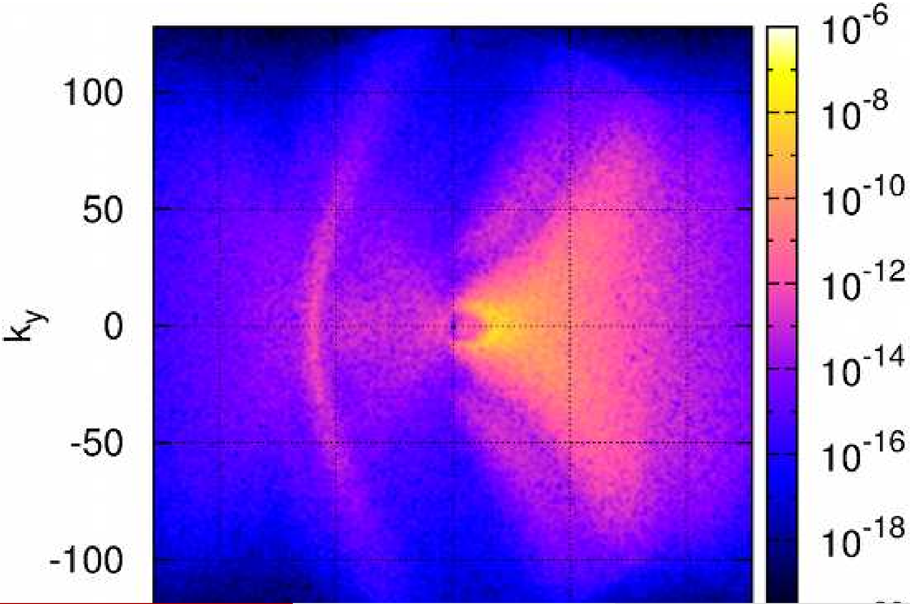
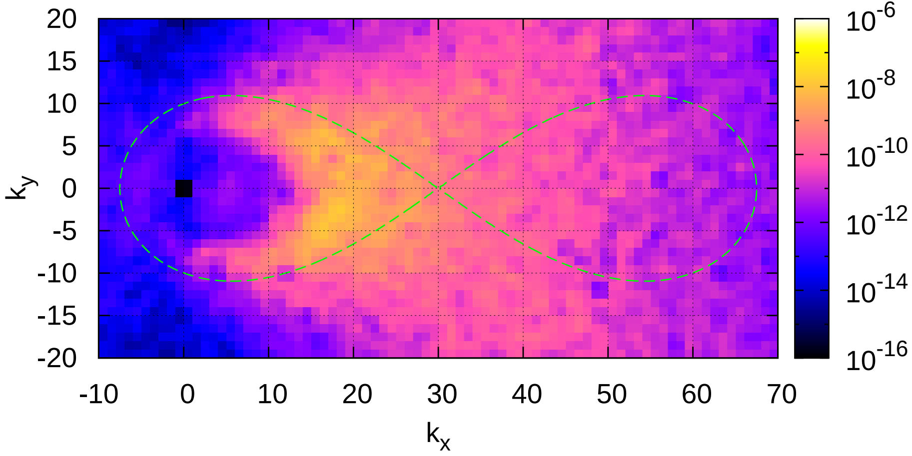
$k_y$

# $k_0 = 30.\ \ T = 1204\,T_0.$



# $k_0 = 30. \quad T = 1204\,T_0.$



# Standing wave instability, general case.

Very interesting and instructive is the case of interaction of two waves $a_{\vec{k}_0}$ and $a_{-\vec{k}_0}$ in the presence of a 4-waves interaction. In this case resulting waves $\vec{k}_3$ and $\vec{k}_4$ has to obey the following relation

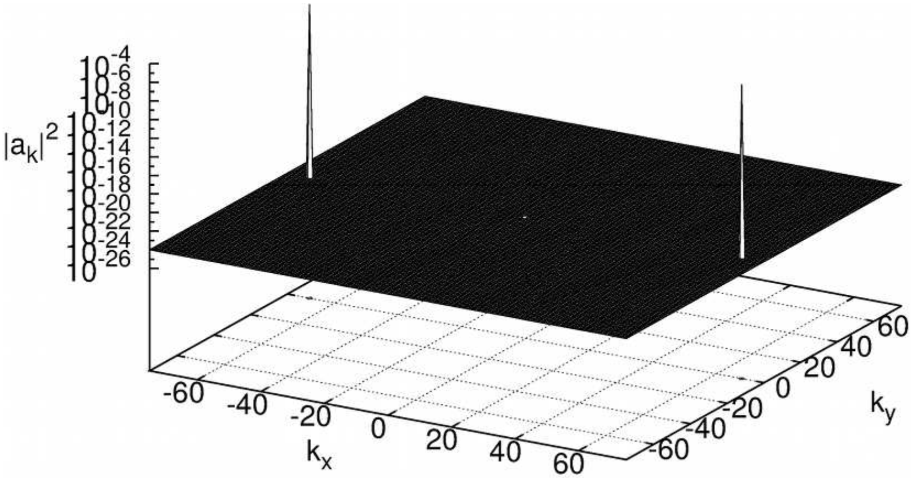
$$\vec{k}_0 + (-\vec{k}_0) = \vec{0} = \vec{k}_3 + \vec{k}_4, \Rightarrow \vec{k}_3 = -\vec{k}_4.$$

If we have a dispersion relation depending only on the magnitude of the wavevector, the condition on the resonance of the frequencies gives us

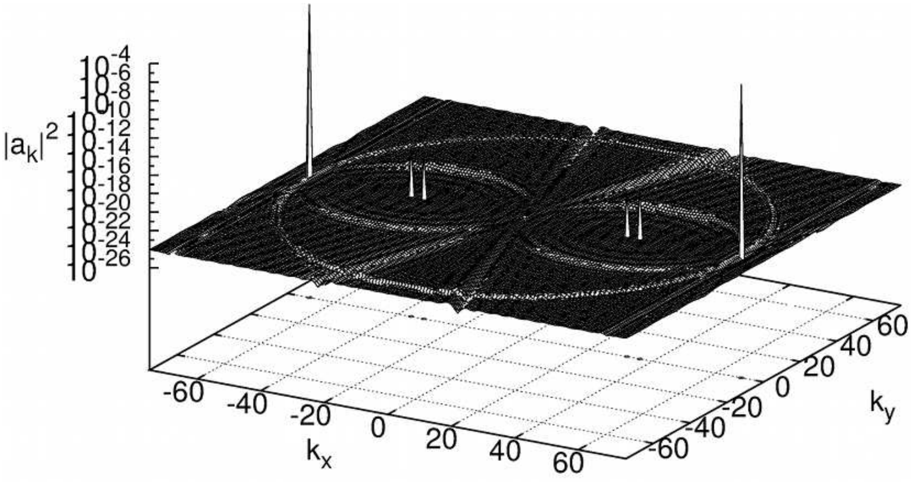
$$2\omega_{k_0} = 2\omega_{k_3},$$

which in case of capillary and gravity waves results in $|\vec{k}_3| = |\vec{k}_0|$, with arbitrary direction.
In other words resonant curve is a circle with the centre at zero wave number vector and of radius $|\vec{k}_0|$. It is clear that such a process is general for any isotropic dispersion.

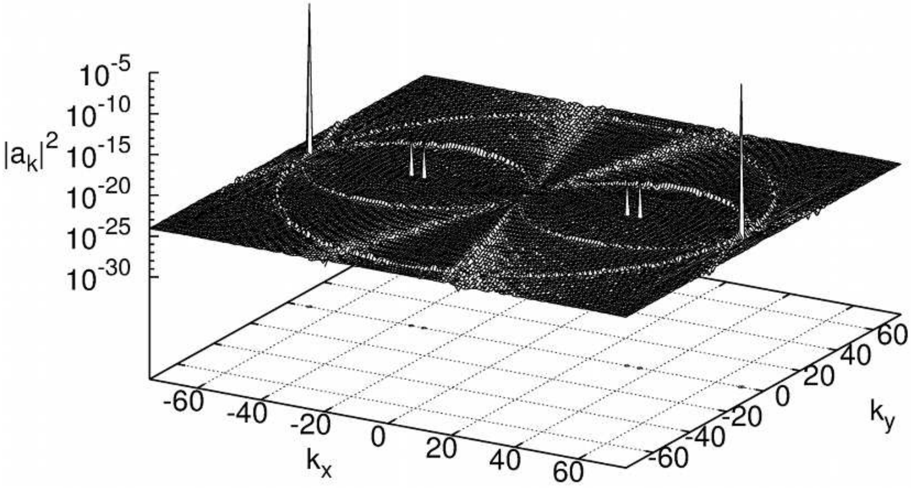
## $k_0 = 68.$ $T = 0.$


$|a_k|^2$
10-4
10-6
10-8
10-10
10-12
10-14
10-16
10-18
10-20
10-22
10-24
10-26
10
-60
-40
-20
0
20
40
60
$k_x$
-60
-40
-20
0
20
40
60
$k_y$

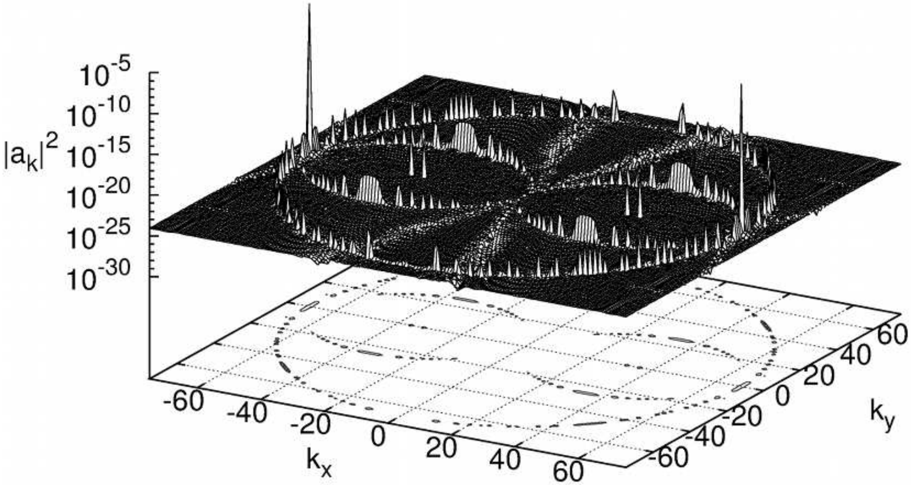
$k_0 = 68.\ \ T = 14T_0.$


$|a_k|^2$
$10^{-4}$
$10^{-6}$
$10^{-8}$
$10^{-10}$
$10^{-12}$
$10^{-14}$
$10^{-16}$
$10^{-18}$
$10^{-20}$
$10^{-22}$
$10^{-24}$
$10^{-26}$
-60 -40 -20 0 20 40 60
$k_x$
-60 -40 -20 0 20 40 60
$k_y$

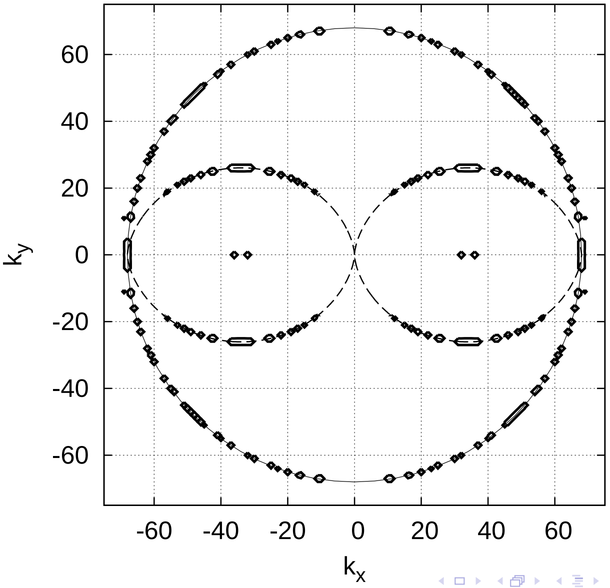
# $k_0 = 68. \ T = 57 T_0.$



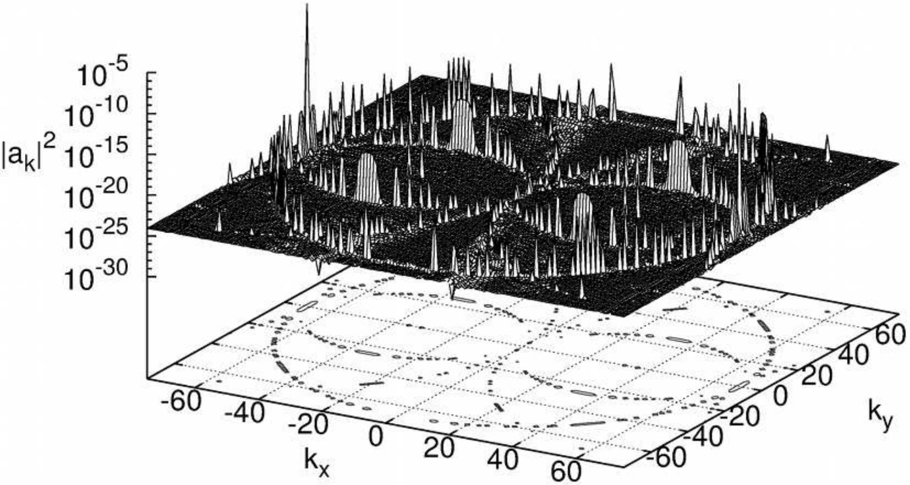
$k_0 = 68.\ \ T = 283\,T_0.$


$|a_k|^2$
$10^{-5}$
$10^{-10}$
$10^{-15}$
$10^{-20}$
$10^{-25}$
$10^{-30}$
-60
-40
-20
0
20
40
60
$k_x$
60
40
20
0
-20
-40
-60
$k_y$

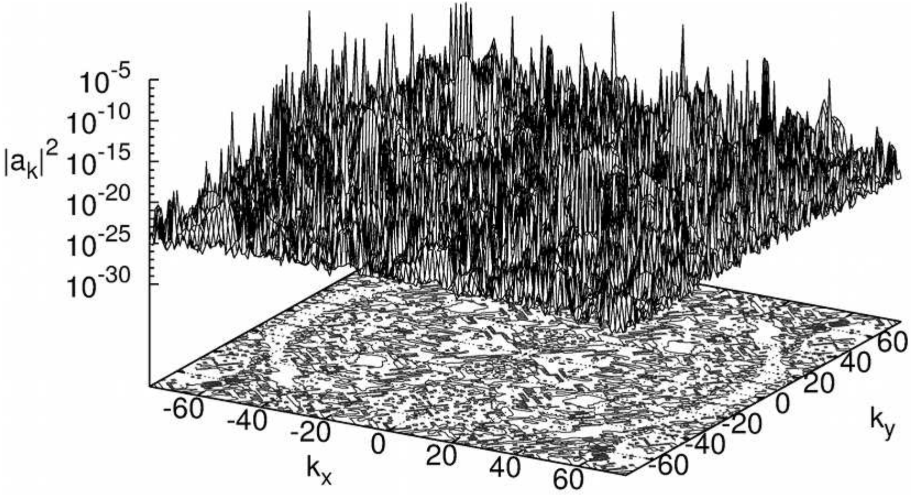
# $k_0 = 68.\ T = 283\,T_0.$



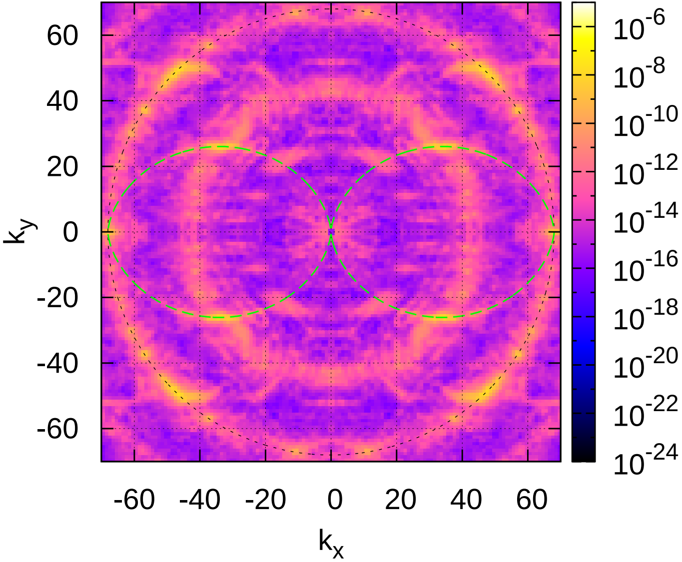
# $k_0 = 68.\ \ T = 509\,T_0.$



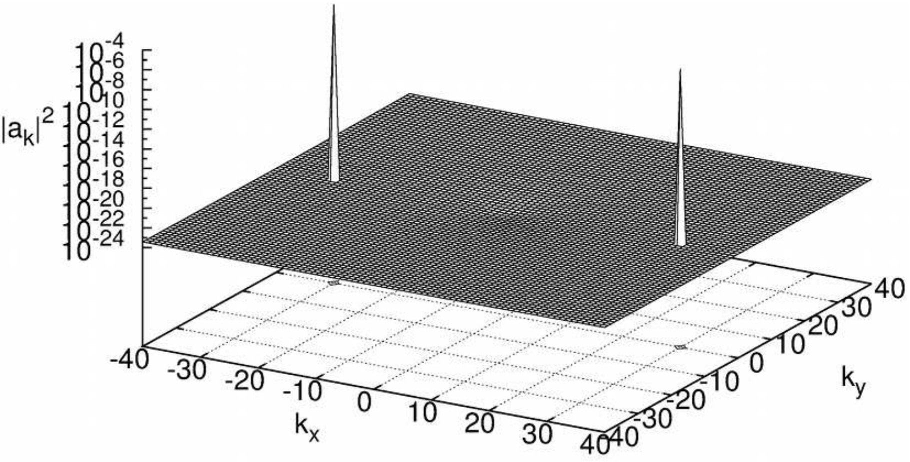
# $k_0 = 68.\ \ T = 1018\,T_0.$



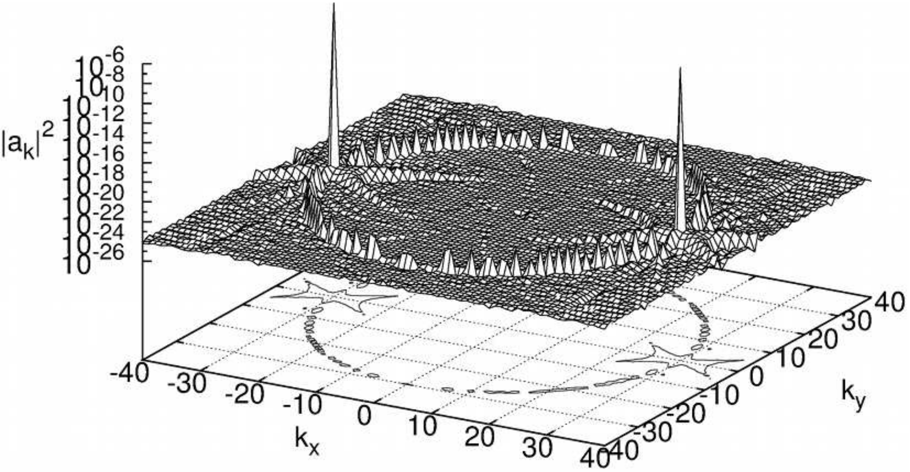
# $k_0 = 68.\ \ T = 2587\,T_0.$



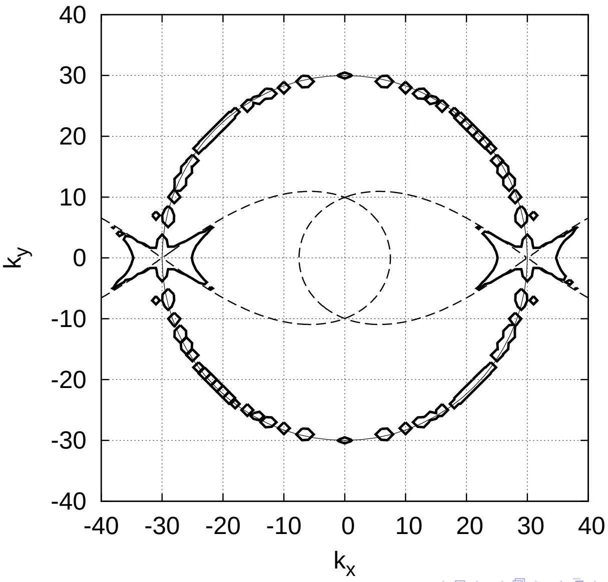
## $k_0 = 30.\ \ T = 0.$


$|a_k|^2$
10-4
10-6
10-8
10-10
10-12
10-14
10-16
10-18
10-20
10-22
10-24
10
-40
-30
-20
-10
0
10
20
30
40
$k_x$
-40
-30
-20
-10
0
10
20
30
40
$k_y$

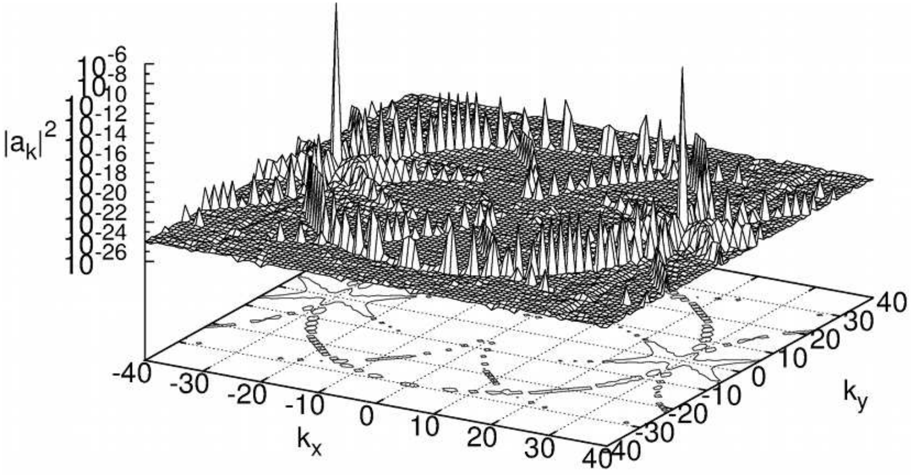
# $k_0 = 30.\ \ T = 116\, T_0.$


$|a_k|^2$
$10^{-6}$
$10^{-8}$
$10^{-10}$
$10^{-12}$
$10^{-14}$
$10^{-16}$
$10^{-18}$
$10^{-20}$
$10^{-22}$
$10^{-24}$
$10^{-26}$
$10$
-40 -30 -20 -10 0 10 20 30 40
$k_x$
-40 -30 -20 -10 0 10 20 30 40
$k_y$

$k_0 = 30.\ \ T = 116\,T_0.$

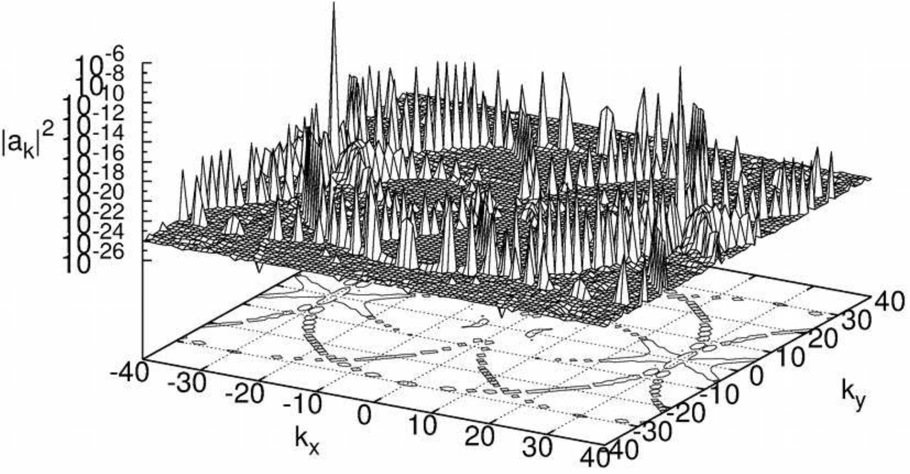


# $k_0 = 30.\ \ T = 232\,T_0.$

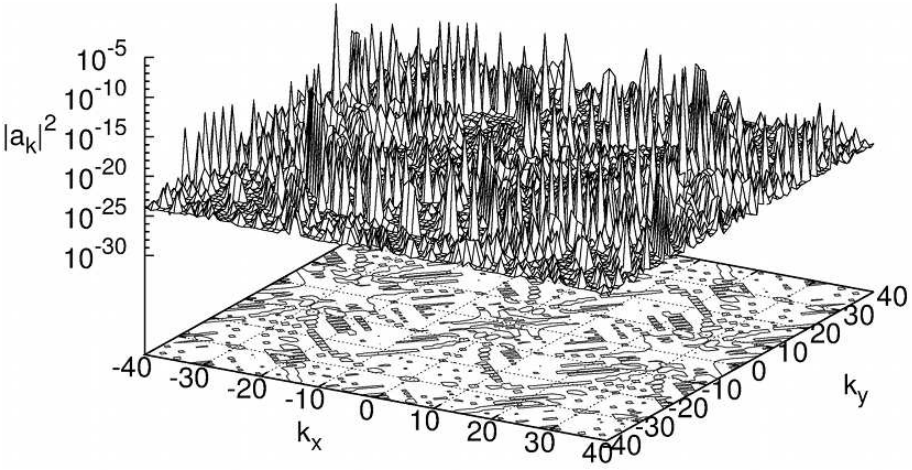


$|a_k|^2$
$10^{-6}$
$10^{-8}$
$10^{-10}$
$10^{-12}$
$10^{-14}$
$10^{-16}$
$10^{-18}$
$10^{-20}$
$10^{-22}$
$10^{-24}$
$10^{-26}$
-40 -30 -20 -10 0 10 20 30 40
$k_x$
-40 -30 -20 -10 0 10 20 30 40
$k_y$

## $k_0 = 30.\ \ T = 348\,T_0.$

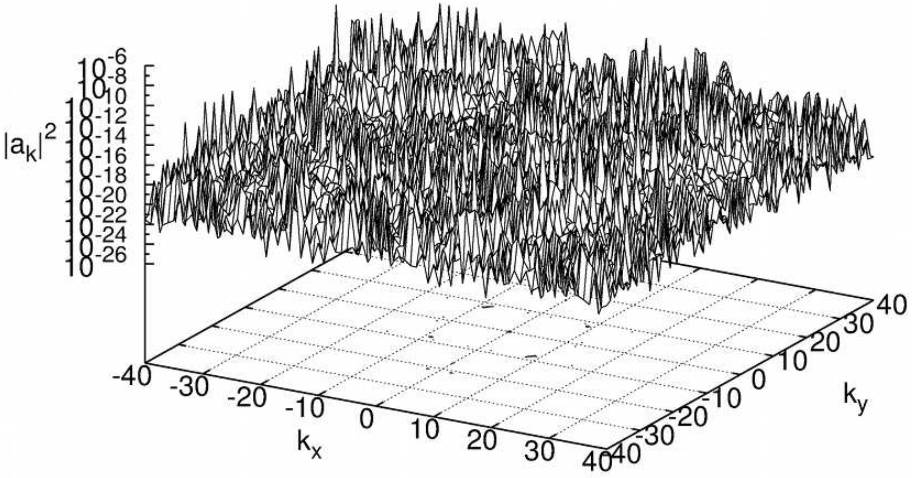

$|a_k|^2$
$10^{-6}$
$10^{-8}$
$10^{-10}$
$10^{-12}$
$10^{-14}$
$10^{-16}$
$10^{-18}$
$10^{-20}$
$10^{-22}$
$10^{-24}$
$10^{-26}$
-40 -30 -20 -10 0 10 20 30 40
$k_x$
-40 -30 -20 -10 0 10 20 30 40
$k_y$

# $k_0 = 30.\ \ T = 463\,T_0.$

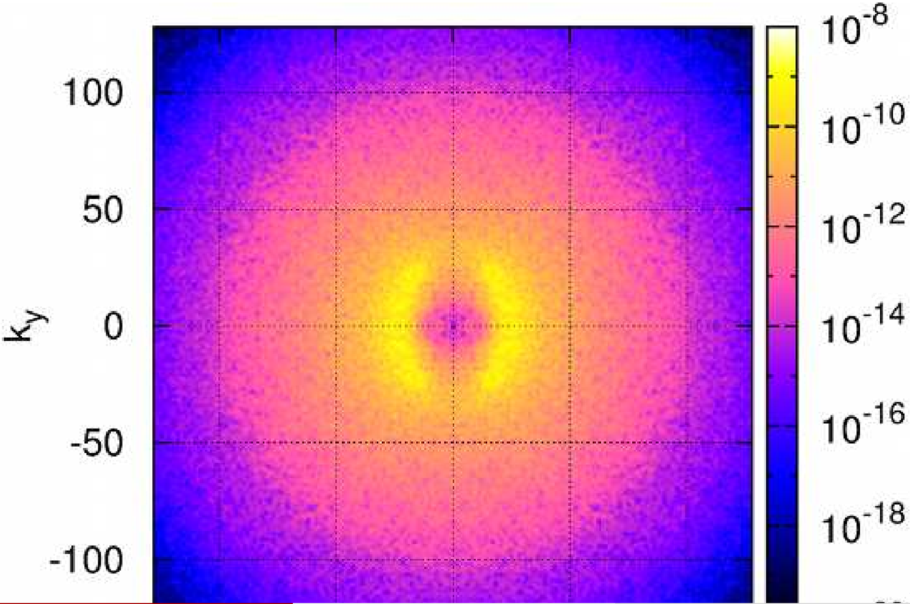


$|a_k|^2$
$10^{-5}$
$10^{-10}$
$10^{-15}$
$10^{-20}$
$10^{-25}$
$10^{-30}$
-40
-30
-20
-10
0
10
20
30
40
$k_x$
-40
-30
-20
-10
0
10
20
30
40
$k_y$

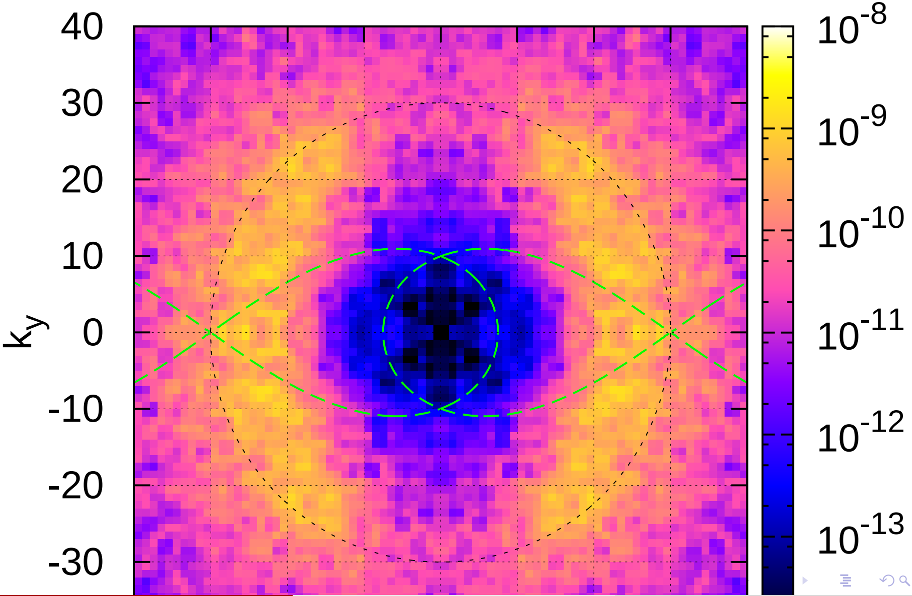
## $k_0 = 30.\ \ T = 580\,T_0.$


$|a_k|^2$
$10^{-6}$
$10^{-8}$
$10^{-10}$
$10^{-12}$
$10^{-14}$
$10^{-16}$
$10^{-18}$
$10^{-20}$
$10^{-22}$
$10^{-24}$
$10^{-26}$
$k_x$
-40 -30 -20 -10 0 10 20 30 40
$k_y$
-40 -30 -20 -10 0 10 20 30 40

## $k_0 = 30. \; T = 3068\, T_0.$



## $k_0 = 30.\ \ T = 3068\,T_0.$



# Results and open questions.

- Performed simulation of resonant interactions on the discrete grid.
- Observed stochastization of the wave field.
- New type of instability in the presence of 4-wave interaction.
- Possible way of isotropic excitation in wave-tank experiments.

KAO, *"Numerical Simulation of Weak Turbulence of Surface Waves"*, PhD thesis, Landau Institute, (2003)
Dyachenko, KAO, Zakharov, *"Decay of the monochromatic capillary wave"*, JETP Lett., 77, 9, 477-481 (2003) arXiv:physics/0308100
KAO, Dyachenko, Zakharov, *"Numerical simulation of surface waves instability on a discrete grid"*, arXiv: 1212.2225.
All these texts can be found at `http://math.unm.edu/~alexkor/`