Local Hamiltonian Structure of Three-Dimensional Benney System

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Three-Dimensional Benney System

- Recently (2006) D.J. Benney considered a three-dimensional motion of a finite depth fluid. Corresponding nonlinear system in partial derivatives is written on three components of the velocity $u(x,y,z,t), v(x,y,z,t), w(x,y,z,t)$ and the profile $\eta(x,z,t)$ of a free surface:

$$u_x + v_y + w_z = 0,$$

$$u_t + uu_x + vu_y + wu_z = -\eta_x,$$

$$w_t + uw_x + vw_y + ww_z = -\eta_z$$

with the boundary conditions

$$v = 0, \quad y = 0,$$

$$\eta_t + u\eta_x + w\eta_z - v = 0, \quad y = \eta.$$

Moments and the Hydrodynamic Lattice

Introducing infinitely many moments

$$A^{k,m}(x,z,t)=\int_0^{\eta}u^k(x,y,z,t)w^m(x,y,z,t)dy,\quad k,m=0,1,...$$

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$$A_t^{k,m}+A_x^{k+1,m}+A_z^{k,m+1}+kA^{k-1,m}A_x^{0,0}+mA^{k,m-1}A_z^{0,0}=0,$$

which has just four local conservation laws, i.e.

$$A_t^{0,0}+A_x^{1,0}+A_z^{0,1}=0,$$

$$A_t^{1,0}+\left(A^{2,0}+\frac{1}{2}(A^{0,0})^2\right)_x+A_z^{1,1}=0,$$

$$A_t^{0,1}+A_x^{1,1}+\left(A^{0,2}+\frac{1}{2}(A^{0,0})^2\right)_z=0,$$

$$(A^{0,2}+A^{2,0}+(A^{0,0})^2)_t$$

$$+(A^{3,0}+A^{1,2}+2A^{1,0}A^{0,0})_x+(A^{2,1}+A^{0,3}+2A^{0,1}A^{0,0})_z=0.$$

Moments and the Vlasov Kinetic Equation

- Now we introduce the Vlasov kinetic equation

$$f_t + pf_x + qf_z - f_p A_x^{0,0} - f_q A_z^{0,0} = 0.$$

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Vlasov kinetic equation implies again the Benney hydrodynamic lattice
$$A_t^{k,m} + A_x^{k+1,m} + A_z^{k,m+1} + kA^{k-1,m}A_x^{0,0} + mA^{k,m-1}A_z^{0,0} = 0.$$

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Vlasov kinetic equation possesses the Hamiltonian structure

$$f_t = \{f, H\}_{\rm LP},$$

where the standard (ultralocal) Lie–Poisson bracket

$$\{f, g\}_{\rm LP} := f_p g_x + f_q g_z - f_x g_p - f_z g_q,$$

and the Hamiltonian function $H = \frac{1}{2}(p^2 + q^2) + A^{0,0}$.

The Vlasov Kinetic Equation

- Introducing the functional

$$\mathbf{H} = \frac{1}{2} \int \int [A^{0,2} + A^{2,0} + (A^{0,0})^2] dxdz$$

such that $H = \delta \mathbf{H} / \delta f$, the Vlasov kinetic equation also can be written in the theoretic-field Hamiltonian form

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$$f_t=\{f,\mathbf{H}\},$$

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$$\{f,\mathbf{H}\}:=(f_p\partial_x+f_q\partial_z-f_x\partial_p-f_z\partial_q)\frac{\delta\mathbf{H}}{\delta f}.$$

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Indeed,

$$\frac{\delta\mathbf{H}}{\delta f}=\frac{1}{2}\frac{\delta A^{0,2}}{\delta f}+\frac{1}{2}\frac{\delta A^{2,0}}{\delta f}+A^{0,0}\frac{\delta A^{0,0}}{\delta f}=\frac{1}{2}(p^2+q^2)+A^{0,0}\equiv H,$$

where we utilized the elementary variational property

$$\frac{\delta A^{k,m}}{\delta f}=p^kq^m.$$

The Vlasov Kinetic Equation

- **Main result** is: *Benney hydrodynamic lattice*

$$A_t^{k,m} + A_x^{k+1,m} + A_z^{k,m+1} + kA^{k-1,m}A_x^{0,0} + mA^{k,m-1}A_z^{0,0} = 0$$

has a local Hamiltonian structure

$$A_t^{i,j} = -[iA^{i+k-1,j+m}\partial_x + k\partial_x A^{i+k-1,j+m}$$

$$+jA^{i+k,j+m-1}\partial_z + m\partial_z A^{i+k,j+m-1}]\frac{\delta \mathbf{H}}{\delta A^{k,m}},$$

where the Hamiltonian is

$$\mathbf{H} = \frac{1}{2}\int\int[A^{0,2} + A^{2,0} + (A^{0,0})^2]dxdz.$$

Indeed, in a general case

$$\mathbf{H} = \int\int h(A^{0,0}, A^{0,1}, A^{1,0}, A^{0,2}, A^{1,1}, A^{2,0}, ...)dxdz$$

and

$$\frac{\delta \mathbf{H}}{\delta f} = \frac{\partial h}{\partial A^{k,m}} \frac{\delta A^{k,m}}{\delta f} = p^k q^m \frac{\partial h}{\partial A^{k,m}}.$$

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Then multiplying

$$f_t = \{f, \mathbf{H}\} = (f_p \partial_x + f_q \partial_z - f_x \partial_p - f_z \partial_q) \frac{\delta \mathbf{H}}{\delta f}$$

by $p^k q^m$ and integrating over p and q, one can see that

$$A_t^{i,j} = \int\int p^i q^j f_t dpdq = \int\int p^i q^j \{f, \mathbf{H}\} dpdq$$

$$= \int\int p^i q^j \left[f_p \left(\frac{\delta \mathbf{H}}{\delta f}\right)_x + f_q \left(\frac{\delta \mathbf{H}}{\delta f}\right)_z - f_x \left(\frac{\delta \mathbf{H}}{\delta f}\right)_p - f_z \left(\frac{\delta \mathbf{H}}{\delta f}\right)_q \right] dpdq$$

The Vlasov Kinetic Equation

$$= \left(\frac{\partial h}{\partial A^{k,m}}\right)_x \int\int p^{k+i} q^{m+j} f_p dpdq + \left(\frac{\partial h}{\partial A^{k,m}}\right)_z \int\int p^{k+i} q^{m+j} f_q dpdq$$

$$-k\frac{\partial h}{\partial A^{k,m}} \int\int p^{k+i-1} q^{m+j} f_x dpdq - m\frac{\partial h}{\partial A^{k,m}} \int\int p^{k+i} q^{m+j-1} f_z dpdq,$$

then integrating by parts two first integrals (with respect to p and q, respectively), we obtain

$$= -(k+i)A^{k+i-1,m+j}\left(\frac{\partial h}{\partial A^{k,m}}\right)_x - (m+j)A^{k+i,m+j-1}\left(\frac{\partial h}{\partial A^{k,m}}\right)_z$$

$$-k\frac{\partial h}{\partial A^{k,m}} A_x^{k+i-1,m+j} - m\frac{\partial h}{\partial A^{k,m}} A_z^{k+i,m+j-1}.$$

This is nothing but precisely

$$A_t^{i,j} = -[iA^{i+k-1,j+m}\partial_x + k\partial_x A^{i+k-1,j+m}$$

$$+jA^{i+k,j+m-1}\partial_z + m\partial_z A^{i+k,j+m-1}]\frac{\delta \mathbf{H}}{\delta A^{k,m}}.$$

Two-Dimensional Poisson Bracket

- In the above particular case this Hamiltonian system yields Benney hydrodynamic lattice. Corresponding two-dimensional Poisson bracket

$$\{A^{i,j}(x,z),A^{k,m}(x',z')\}=[(i+k)A^{i+k-1,j+m}\partial_x+kA_x^{i+k-1,j+m}$$
$$+(j+m)A^{i+k,j+m-1}\partial_z+mA_z^{i+k,j+m-1}]\delta(x-x')\delta(z-z') \quad (1)$$

we call the *two-dimensional* Kupershmidt–Manin Poisson bracket, because earlier the *one-dimensional* Kupershmidt–Manin Poisson bracket

$$\{A^k(x),A^m(x')\}=[(k+m)A^{k+m-1}\partial_x+mA_x^{k+m-1}]\delta(x-x')$$

was derived for the one-dimensional Benney hydrodynamic chain

$$A_t^k+A_x^{k+1}+kA^{k-1}A_x^0=0,\ \ k=0,1,\ldots$$

Poisson bracket (1) is a *first* example of *two-dimensional* differential-geometric Poisson brackets of a first order generalised to an *infinitely* many component case.

Local Conservation Laws

- D.J. Benney was able to find just four local conservation laws. Actually, explanation of this result is based on existence of the local Hamiltonian structure. Indeed, any Hamiltonian system possesses just four local conservation laws for an arbitrary Hamiltonian density:
- the continuity conservation law

$$A_t^{0,0} + \left(kA^{k-1,m}\frac{\partial h}{\partial A^{k,m}}\right)_x + \left(mA^{k,m-1}\frac{\partial h}{\partial A^{k,m}}\right)_z = 0;$$

- the conservation law of the momentum $(x, z$ components)

$$A_t^{1,0} + \left[(k+1)A^{k,m}\frac{\partial h}{\partial A^{k,m}} - h\right]_x + \left(mA^{k+1,m-1}\frac{\partial h}{\partial A^{k,m}}\right)_z = 0,$$

$$A_t^{0,1} + \left(kA^{k-1,m+1}\frac{\partial h}{\partial A^{k,m}}\right)_x + \left[(m+1)A^{k,m}\frac{\partial h}{\partial A^{k,m}} - h\right]_z = 0;$$

- the conservation law of the energy

$$h_t + \left(kA^{i+k-1,j+m}\frac{\partial h}{\partial A^{i,j}}\frac{\partial h}{\partial A^{k,m}}\right)_x + \left(mA^{i+k,j+m-1}\frac{\partial h}{\partial A^{i,j}}\frac{\partial h}{\partial A^{k,m}}\right)_z = 0.$$

Multi-Component Reductions

- Investigation of infinitely many component quasilinear systems of a first order (i.e. hydrodynamic lattices) is very complicated problem. One of most effective tools is a method of multi-component hydrodynamic reductions. In a general case hydrodynamic reductions can be extracted utilizing *generalized* functions like the Heaviside step-function or the Dirac delta-function.

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- For example, the so called "cold plasma" approximation ansatz (also well known as the "multi-flow" ansatz)

$$f(x,z,p,q,t)=\sum_{k=1}^{N}\eta^k(x,z,t)\delta(p-u^k(x,z,t))\delta(q-w^k(x,z,t))$$

reduces Benney hydrodynamic lattice

$$A_t^{k,m}+A_x^{k+1,m}+A_z^{k,m+1}+kA^{k-1,m}A_x^{0,0}+mA^{k,m-1}A_z^{0,0}=0$$

Multi-Component Reductions

- to the finite component form

$$\eta_t^k + (u^k \eta^k)_x + (w^k \eta^k)_z = 0,$$

$$u_t^k + u^k u_x^k + w^k u_z^k + \left(\sum_{m=1}^{N} \eta^m \right)_x = 0,$$

$$w_t^k + u^k w_x^k + w^k w_z^k + \left(\sum_{m=1}^{N} \eta^m \right)_z = 0,$$

where all moments are expressed polynomially via new field variables η^k, u^s, w^m

$$A^{k,m} = \sum_{p=1}^{N} \eta^p (u^p)^k (w^p)^m.$$

Multi-Component Reductions

- In the simplest case $N = 1$, this is nothing but a system describing the irrotational two-dimensional hydrodynamics

$$\eta_t + (u\eta)_x + (w\eta)_z = 0,$$

$$u_t + uu_x + wu_z + \eta_x = 0, \;\; w_t + uw_x + ww_z + \eta_z = 0.$$

- This system also can be obtained directly from two-dimensional Benney system by the reduction $u(x, z, t), w(x, z, t)$ and $v = -y(u_x + w_z)$.

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- This system also can be obtained directly from two-dimensional Benney system by the reduction $u(x, z, t), w(x, z, t)$ and $v = -y(u_x + w_z)$.
- This system possesses a local Hamiltonian structure, where corresponding Poisson bracket is (here we write just nonzero components)

$$\{\eta(x, z), u(x', z')\} = \{u(x, z), \eta(x', z')\} = \delta'(x - x')\delta(z - z'),$$

$$\{\eta(x, z), w(x, z)\} = \{w(x, z), \eta(x', z')\} = \delta(x - x')\delta'(z - z'),$$

$$\{u(x, z), w(x', z')\} = \text{-}\{w(x, z), u(x', z')\} = \frac{u_z\text{-}w_x}{\eta}\delta(x - x')\delta(z - z').$$

Indeed, the substitution $A^{k,m} = \eta u^k w^m$ into the two-dimensional Kupershmidt–Manin Poisson bracket

$$A_t^{i,j} = -[iA^{i+k-1,j+m}\partial_x + k\partial_x A^{i+k-1,j+m}$$

$$+jA^{i+k,j+m-1}\partial_z + m\partial_z A^{i+k,j+m-1}]\frac{\delta \mathbf{H}}{\delta A^{k,m}}.$$

implies above two-dimensional Poisson bracket

$$\{\eta(x,z), u(x',z')\} = \{u(x,z), \eta(x',z')\} = \delta'(x-x')\delta(z-z'),$$

$$\{\eta(x,z), w(x,z)\} = \{w(x,z), \eta(x',z')\} = \delta(x-x')\delta'(z-z'),$$

$$\{u(x,z), w(x',z')\} = \text{-}\{w(x,z), u(x',z')\} = \frac{u_z\text{-}w_x}{\eta}\delta(x-x')\delta(z-z').$$

Multi-Component Reductions

- This system possesses also just four local conservation laws

$$\eta_t + (u\eta)_x + (w\eta)_z = 0,$$

$$(u\eta)_t + \left(u^2\eta + \frac{1}{2}\eta^2\right)_x + (uw\eta)_z = 0,$$

$$(w\eta)_t + (uw\eta)_x + \left(w^2\eta + \frac{1}{2}\eta^2\right)_z = 0,$$

$$[(u^2+w^2)\eta+\eta^2]_t+[(u^2+w^2)u\eta+2u\eta^2]_x+[(u^2+w^2)w\eta+2w\eta^2]_z = 0.$$

Integrability

- Benney hydrodynamic lattice

$$A_t^{k,m} + A_x^{k+1,m} + A_z^{k,m+1} + kA^{k-1,m}A_x^{0,0} + mA^{k,m-1}A_z^{0,0} = 0$$

as well as its multi component reduction

$$\eta_t^k + (u^k\eta^k)_x + (w^k\eta^k)_z = 0,$$

$$u_t^k + u^k u_x^k + w^k u_z^k + \left(\sum_{m=1}^{N}\eta^m\right)_x = 0,$$

$$w_t^k + u^k w_x^k + w^k w_z^k + \left(\sum_{m=1}^{N}\eta^m\right)_z = 0$$

possess just four local conservation laws. Thus these nonlinear systems are non-integrable.