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Riemann-Hilbert Problems and Soliton equations. The Reduction problem and Hamiltonian properties.

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It is my pleasure to congratulate Vladimir Evgen'evich for his 75-th birthday!

PLAN

- RHP with canonical normalization
- Jets of order k
- Reductions of polynomial bundles
- On N-wave equations – $k = 1$
- New N-wave equations – $k \geq 2$
- a family of mKdV equations related to $so(8)$ simple Lie algebra
- Hamiltonian properties of N-wave equations
- Conclusions and open questions

Based on:

- V. S. Gerdjikov, D. J. Kaup. *Reductions of* 3×3 *polynomial bundles and new types of integrable* 3*–wave interactions.* In Nonlinear evolution equations: integrability and spectral methods, Ed. A. P. Fordy, A. Degasperis, M. Lakshmanan, Manchester University Press, (1981), p. 373–380

- V. S. Gerdjikov. On new types of integrable 4-wave interactions. AIP Conf. proc. **1487** pp. 272-279; (2012).

- V. S. Gerdjikov. Riemann-Hilbert Problems with canonical normalization and families of commuting operators. Pliska Stud. Math. Bulgar. **21**, 201–216 (2012).

- V. S. Gerdjikov. Derivative Nonlinear Schrödinger Equations with $\mathbb{Z}_N$ and $\mathbb{D}_N$–Reductions. Romanian Journal of Physics, **58**, Nos. 5-6, 573-582 (2013).

- V. S. Gerdjikov, A. B. Yanovski *On soliton equations with* $\mathbb{Z}_h$

and $\mathbb{D}_h$ *reductions: conservation laws and generating operators.* J. Geom. Symmetry Phys. **31**, 57–92 (2013).

- V. S. Gerdjikov, A B Yanovski. Riemann-Hilbert Problems, families of commuting operators and soliton equations Journal of Physics: Conference Series **482** (2014) 012017 doi:10.1088/1742-6596/482/1/012017

RHP with canonical normalization

$$\xi^{+}(x,t,\lambda) = \xi^{-}(x,t,\lambda) G(x,t,\lambda), \qquad \lambda^k \in \mathbb{R}, \qquad \lim_{\lambda\to\infty} \xi^{+}(x,t,\lambda) = \mathbb{1},$$

$\xi^{\pm}(x,t,\lambda) \in \mathfrak{G}$

Consider particular type of dependence $G(x,t,\lambda)$:

$$i\frac{\partial G}{\partial x} - \lambda^k [J, G(x,t,\lambda)] = 0, \qquad i\frac{\partial G}{\partial t} - \lambda^k [K, G(x,t,\lambda)] = 0.$$

where $J \in \mathfrak{h} \subset \mathfrak{g}$.

The canonical normalization of the RHP:

$$\xi^{\pm}(x,t,\lambda) = \exp Q(x,t,\lambda), \qquad Q(x,t,\lambda) = \sum_{k=1}^{\infty} Q_k(x,t)\lambda^{-k}.$$

where all $Q_k(x,t) \in \mathfrak{g}$. However,

$$\mathcal{J}(x,t,\lambda) = \xi^{\pm}(x,t,\lambda) J \hat{\xi}^{\pm}(x,t,\lambda), \qquad \mathcal{K}(x,t,\lambda) = \xi^{\pm}(x,t,\lambda) K \hat{\xi}^{\pm}(x,t,\lambda),$$

belong to the algebra $\mathfrak{g}$ for any J and K from $\mathfrak{g}$. If in addition K also belongs to the Cartan subalgebra $\mathfrak{h}$, then

$$[\mathcal{J}(x,t,\lambda), \mathcal{K}(x,t,\lambda)] = 0.$$

Zakharov-Shabat theorem

Theorem 1. *Let $\xi^{\pm}(x,t,\lambda)$ be solutions to the RHP whose sewing function depends on the auxiliary variables x and t as above. Then $\xi^{\pm}(x,t,\lambda)$ are fundamental solutions of the following set of differential operators:*

$$\begin{aligned} L\xi^{\pm} \equiv & i\frac{\partial \xi^{\pm}}{\partial x} + U_s(x,t,\lambda)\xi^{\pm}(x,t,\lambda) - \lambda^k [J, \xi^{\pm}(x,t,\lambda)] = 0, \\ M\xi^{\pm} \equiv & i\frac{\partial \xi^{\pm}}{\partial t} + V(x,t,\lambda)\xi^{\pm}(x,t,\lambda) - \lambda^k [K, \xi^{\pm}(x,t,\lambda)] = 0. \end{aligned}$$

Proof. Introduce the functions:

$$\begin{aligned} g^{\pm}(x,t,\lambda) &= i\frac{\partial \xi^{\pm}}{\partial x}\hat{\xi}^{\pm}(x,t,\lambda) + \lambda^k \xi^{\pm}(x,t,\lambda) J \hat{\xi}^{\pm}(x,t,\lambda), \\ p^{\pm}(x,t,\lambda) &= i\frac{\partial \xi^{\pm}}{\partial t}\hat{\xi}^{\pm}(x,t,\lambda) + \lambda^k \xi^{\pm}(x,t,\lambda) K \hat{\xi}^{\pm}(x,t,\lambda), \end{aligned}$$

and using

$$i\frac{\partial G}{\partial x} - \lambda^k [J, G(x,t,\lambda)] = 0, \qquad i\frac{\partial G}{\partial t} - \lambda^k [K, G(x,t,\lambda)] = 0.$$

prove that

$$g^+(x,t,\lambda) = g^-(x,t,\lambda), \qquad p^+(x,t,\lambda) = p^-(x,t,\lambda),$$

which means that these functions are analytic functions of λ in the whole complex λ-plane. Next we find that:

$$\lim_{\lambda\to\infty} g^+(x,t,\lambda) = \lambda^k J, \qquad \lim_{\lambda\to\infty} p^+(x,t,\lambda) = \lambda^k K.$$

and make use of Liouville theorem to get

$$g^+(x,t,\lambda) = g^-(x,t,\lambda) = \lambda^k J - \sum_{l=1}^{k} U_{s;l}(x,t)\lambda^{k-l},$$

$$p^+(x,t,\lambda) = p^-(x,t,\lambda) = \lambda^k K - \sum_{l=1}^{k} V_l(x,t)\lambda^{k-l}.$$

We shall see below that the coefficients $U_l(x,t)$ and $V_l(x,t)$ can be expressed in terms of the asymptotic coefficients Q_s of $\xi^{\pm}(x,t,\lambda)$.

Now remember the definition of $g^{+}(x,t,\lambda)$

$$
\begin{aligned}
g^{\pm}(x,t,\lambda) &= i\frac{\partial \xi^{\pm}}{\partial x}\hat{\xi}^{\pm}(x,t,\lambda) + \lambda^k \xi^{\pm}(x,t,\lambda) J \hat{\xi}^{\pm}(x,t,\lambda) \\
&= \lambda^k J - \sum_{l=1}^{k} U_{s;l}(x,t)\lambda^{k-l},
\end{aligned}
$$

Multiply both sides by $\xi^{\pm}(x,t,\lambda)$ and move all the terms to the left:

$$
i\frac{\partial \xi^{\pm}}{\partial x} + \sum_{l=1}^{k} U_l(x,t)\lambda^{k-l}\xi^{\pm}(x,t,\lambda) - \lambda^k [J, \xi^{\pm}(x,t,\lambda)] = 0,
$$

i.e. $L\xi^{\pm}(x,t,\lambda) = 0$. □

Lemma 1. *The operators L and M commute*

$$
[L, M] = 0,
$$

i.e. the following set of equations hold:

$$i\frac{\partial U}{\partial t} - i\frac{\partial V}{\partial x} + [U(x,t,\lambda) - \lambda^k J, V(x,t,\lambda) - \lambda^k K] = 0.$$

where

$$U(x,t,\lambda) = \sum_{l=1}^{k} U_l(x,t)\lambda^{k-l}, \qquad V(x,t,\lambda) = \sum_{l=0}^{k} V_l(x,t)\lambda^{k-l}.$$

Jets of order k

How to parametrize $U(x,t,\lambda)$ and $V(x,t,\lambda)$?
Use:

$$\xi^{\pm}(x,t,\lambda) = \exp Q(x,t,\lambda), \qquad Q(x,t,\lambda) = \sum_{k=1}^{\infty} Q_k(x,t)\lambda^{-k}.$$

and consider the jets of order k of $\mathcal{J}(x,\lambda)$ and $\mathcal{K}(x,\lambda)$:

$$\mathcal{J}(x,t,\lambda) \equiv \left(\lambda^k \xi^{\pm}(x,t,\lambda) J_l \hat{\xi}^{\pm}(x,t,\lambda)\right)_+ = \lambda^k J - U(x,t,\lambda),$$
$$\mathcal{K}(x,t,\lambda) \equiv \left(\lambda^k \xi^{\pm}(x,t,\lambda) K \hat{\xi}^{\pm}(x,t,\lambda)\right)_+ = \lambda^k K - V(x,t,\lambda).$$

Express $U(x) \in \mathfrak{g}$ in terms of $Q_s(x)$:

$$\mathcal{J}(x,t,\lambda) = J + \sum_{k=1}^{\infty} \frac{1}{k!} \mathrm{ad}\,_Q^k J, \qquad \mathcal{K}(x,t,\lambda) = K + \sum_{k=1}^{\infty} \frac{1}{k!} \mathrm{ad}\,_Q^k K,$$
$$\mathrm{ad}\,_Q Z = [Q,Z], \qquad \mathrm{ad}\,_Q^2 Z = [Q,[Q,Z]], \quad \dots$$

and therefore for U_l we get:

$$U_1(x,t) = -\mathrm{ad}\,_{Q_1} J, \qquad U_2(x,t) = -\mathrm{ad}\,_{Q_2} J - \frac{1}{2} \mathrm{ad}\,_{Q_1}^2 J$$
$$U_3(x,t) = -\mathrm{ad}\,_{Q_3} J - \frac{1}{2} \left(\mathrm{ad}\,_{Q_2} \mathrm{ad}\,_{Q_1} + \mathrm{ad}\,_{Q_1} \mathrm{ad}\,_{Q_2}\right) J - \frac{1}{6} \mathrm{ad}\,_{Q_1}^3 J.$$

and similar expressions for $V_l(x,t)$ with J replaced by K.

Reductions of polynomial bundles

$$
\begin{array}{lll}
\text{a)} & A\xi^{+,\dagger}(x,t,\epsilon\lambda^*)\hat{A} = \hat{\xi}^{-}(x,t,\lambda), & AQ^{\dagger}(x,t,\epsilon\lambda^*)\hat{A} = -Q(x,t,\lambda), \\
\text{b)} & B\xi^{+,*}(x,t,\epsilon\lambda^*)\hat{B} = \xi^{-}(x,t,\lambda), & BQ^{*}(x,t,\epsilon\lambda^*)\hat{B} = Q(x,t,\lambda), \\
\text{c)} & C\xi^{+,T}(x,t,-\lambda)\hat{C} = \hat{\xi}^{-}(x,t,\lambda), & CQ^{\dagger}(x,t,-\lambda)\hat{C} = -Q(x,t,\lambda),
\end{array}
$$

where $\epsilon^2 = 1$ and A, B and C are elements of the group $\mathfrak{G}$ such that $A^2 = B^2 = C^2 = \mathbb{1}$. As for the $\mathbb{Z}_N$-reductions we may have:

$$
D\xi^{\pm}(x,t,\omega\lambda)\hat{D} = \xi^{\pm}(x,t,\lambda), \qquad DQ(x,t,\omega\lambda)\hat{D} = Q(x,t,\lambda),
$$

where $\omega^N = 1$ and $D^N = \mathbb{1}$.

On N-wave equations – $k = 1$

Zakharov, Shabat, Manakov (1974)

Lax representation involves two Lax operators linear in λ:

$$
\begin{aligned}
L\xi^{\pm} &\equiv i\frac{\partial \xi^{\pm}}{\partial x} + [J, Q(x,t)]\xi^{\pm}(x,t,\lambda) - \lambda[J, \xi^{\pm}(x,t,\lambda)] = 0, \\
M\xi^{\pm} &\equiv i\frac{\partial \xi^{\pm}}{\partial t} + [K, Q(x,t)]\xi^{\pm}(x,t,\lambda) - \lambda[K, \xi^{\pm}(x,t,\lambda)] = 0.
\end{aligned}
$$

The corresponding equations take the form:

$$
i\left[J, \frac{\partial Q}{\partial t}\right] - i\left[K, \frac{\partial Q}{\partial x}\right] - [[J,Q],[K,Q(x,t)]] = 0
$$

$$
Q(x,t) = \begin{pmatrix} 0 & u_1 & u_3 \\ -v_1 & 0 & u_2 \\ -v_3 & -v_2 & 0 \end{pmatrix}, \qquad \begin{aligned} J &= \mathrm{diag}\,(a_1, a_2, a_3), \\ K &= \mathrm{diag}\,(b_1, b_2, b_3), \end{aligned}
$$

Then the 3-wave equations take the form:

$$\begin{aligned}
\frac{\partial u_1}{\partial t} - \frac{a_1 - a_2}{b_1 - b_2}\frac{\partial u_1}{\partial x} + \kappa\epsilon_1\epsilon_2 u_2^* u_3 = 0,\\
\frac{\partial u_2}{\partial t} - \frac{a_2 - a_3}{b_2 - b_3}\frac{\partial u_2}{\partial x} + \kappa\epsilon_1 u_1^* u_3 = 0,\\
\frac{\partial u_3}{\partial t} - \frac{a_1 - a_3}{b_1 - b_3}\frac{\partial u_3}{\partial x} + \kappa\epsilon_2 u_1^* u_2^* = 0,
\end{aligned}$$

where

$$\kappa = a_1(b_2 - b_3) - a_2(b_1 - b_3) + a_3(b_1 - b_2).$$

New 3-wave equations – $k \geq 2$

Let $\mathfrak{g} = sl(3)$ and

$$Q_1(x,t) = \begin{pmatrix} 0 & u_1 & u_3 \\ -v_1 & 0 & u_2 \\ -v_3 & -v_2 & 0 \end{pmatrix}, \qquad Q_2(x,t) = \begin{pmatrix} q_{11} & w_1 & w_3 \\ -z_1 & q_{22} & w_2 \\ -z_3 & -z_2 & q_{33} \end{pmatrix},$$

Fix up $k = 2$. Then the Lax pair becomes

$$L\xi^{\pm} \equiv i\frac{\partial \xi^{\pm}}{\partial x} + U(x,t,\lambda)\xi^{\pm}(x,t,\lambda) - \lambda^2]J, \xi^{\pm}(x,t,\lambda)] = 0,$$
$$M\xi^{\pm} \equiv i\frac{\partial \xi^{\pm}}{\partial t} + V(x,t,\lambda)\xi^{\pm}(x,t,\lambda) - \lambda^2]K, \xi^{\pm}(x,t,\lambda)] = 0,$$

where

$$U \equiv U_2 + \lambda U_1 = \left([J, Q_2(x)] - \frac{1}{2}[[J, Q_1], Q_1(x)] \right) + \lambda [J, Q_1],$$
$$V \equiv V_2 + \lambda V_1 = \left([K, Q_2(x)] - \frac{1}{2}[[K, Q_1], Q_1(x)] \right) + \lambda [K, Q_1].$$

Impose a $\mathbb{Z}_2$-reduction of type a) with $A = \mathrm{diag}\,(1, \epsilon, 1)$, $\epsilon^2 = 1$. Thus Q_1 and Q_2 get reduced into:

$$Q_1 = \begin{pmatrix} 0 & u_1 & 0 \\ \epsilon u_1^* & 0 & u_2 \\ 0 & \epsilon u_2^* & 0 \end{pmatrix}, \qquad Q_2 = \begin{pmatrix} 0 & 0 & w_3 \\ 0 & 0 & 0 \\ w_3^* & 0 & 0 \end{pmatrix},$$

and we obtain new type of integrable 3-wave equations:

$$
\begin{aligned}
&i(a_1-a_2)\frac{\partial u_1}{\partial t}-i(b_1-b_2)\frac{\partial u_1}{\partial x}+\epsilon\kappa u_2^* u_3+\epsilon\frac{\kappa(a_1-a_2)}{(a_1-a_3)}u_1|u_2|^2=0,\\
&i(a_2-a_3)\frac{\partial u_2}{\partial t}-i(b_2-b_3)\frac{\partial u_2}{\partial x}+\epsilon\kappa u_1^* u_3-\epsilon\frac{\kappa(a_2-a_3)}{(a_1-a_3)}|u_1|^2 u_2=0,\\
&i(a_1-a_3)\frac{\partial u_3}{\partial t}-i(b_1-b_3)\frac{\partial u_3}{\partial x}-\frac{i\kappa}{a_1-a_3}\frac{\partial(u_1u_2)}{\partial x}\\
&\quad+\epsilon\kappa\left(\frac{a_1-a_2}{a_1-a_3}|u_1|^2+\frac{a_2-a_3}{a_1-a_3}|u_2|^2\right)u_1u_2+\epsilon\kappa u_3(|u_1|^2-|u_2|^2)=0,
\end{aligned}
$$

where

$$
\kappa=a_1(b_2-b_3)-a_2(b_1-b_3)+a_3(b_1-b_2),\qquad u_3=w_3+\frac{2a_2-a_1-a_3}{2(a_1-a_3)}u_1u_2.
$$

The diagonal terms in the Lax representation are λ-independent.

Two of them read:

$$i(a_1 - a_2)\frac{\partial |u_1|^2}{\partial t} - i(b_1 - b_2)\frac{\partial |u_1|^2}{\partial x} - \epsilon\kappa(u_1 u_2 u_3^* - u_1^* u_2^* u_3) = 0,$$
$$i(a_2 - a_3)\frac{\partial |u_2|^2}{\partial t} - i(b_2 - b_3)\frac{\partial |u_2|^2}{\partial x} - \epsilon\kappa(u_1 u_2 u_3^* - u_1^* u_2^* u_3) = 0,$$

These relations are satisfied identically as a consequence of the NLEE.

New types of 4-wave interactions

The Lax pair for these new equations will be provided by:

$$L\psi = i\frac{\partial \psi}{\partial x} + (U_2(x,t) + \lambda U_1(x,t) - \lambda^2 J)\psi(x,t,\lambda) = 0,$$
$$M\psi = i\frac{\partial \psi}{\partial t} + (V_2(x,t) + \lambda V_1(x,t) - \lambda^2 K)\psi(x,t,\lambda) = 0,$$

where $U_j(x,t)$ and $V_j(x,t)$ are fast decaying smooth functions taking values in the Lie algebra $so(5)$

$$U_1(x,t) = [J, Q_1(x,t)], \qquad U_2(x,t) = [J, Q_2(x,t)] - \frac{1}{2}\mathrm{ad}\,^2_{Q_1} J,$$
$$V_1(x,t) = [K, Q_1(x,t)], \qquad V_2(x,t) = [K, Q_2(x,t)] - \frac{1}{2}\mathrm{ad}\,^2_{Q_1} K.$$

Here $\mathrm{ad}\,_{Q_1} X \equiv [Q_1(x,t), X]$.

Assume $Q_1(x,t)$ and $Q_2(x,t)$ to be generic elements of $so(5)$:

$$Q_1(x,t) = \sum_{\alpha\in\Delta_+} (q^1_\alpha E_\alpha + p^1_\alpha E_{-\alpha}) + r^1_1 H_{e_1} + r^1_2 H_{e_2},$$

$$Q_2(x,t) = \sum_{\alpha\in\Delta_+} (q^2_\alpha E_\alpha + p^2_\alpha E_{-\alpha}) + r^2_1 H_{e_1} + r^2_2 H_{e_2},$$

$$J = a_1 H_{e_1} + a_2 H_{e_2} = \mathrm{diag}\,(a_1, a_2, 0, -a_2, -a_1),$$
$$K = b_1 H_{e_1} + b_2 H_{e_2} = \mathrm{diag}\,(b_1, b_2, 0, -b_2, -b_1),$$

Next we impose on $Q_1(x,t)$ and $Q_2(x,t)$ the natural reduction

$$B_0 U(x,t,\epsilon\lambda^*)^\dagger B_0^{-1} = U(x,t,\lambda), \qquad B_0 = \mathrm{diag}\,(1,\epsilon,1,\epsilon,1), \quad \epsilon^2 = 1.$$

As a result:

$$B_0(\chi^+(x,t,\epsilon\lambda^*))^\dagger B_0^{-1} = (\chi^-(x,t,\lambda))^{-1}, \qquad B_0(T(t,\epsilon\lambda^*))^\dagger B_0^{-1} = (T(t,\lambda))^{-1},$$

which provide $p^1_\alpha = \epsilon(q^1_\alpha)^*$, $p^2_\alpha = \epsilon(q^2_\alpha)^*$. Then the Lax representation will be a (rather complicated) system of 8 NLEE for the 8 independent matrix elements q^1_α and q^2_α.

However we can impose additional $\mathbb{Z}_2$ reduction condition

$$D\xi^\pm(x,t,-\lambda)\hat{D} = \xi^\pm(x,t,\lambda), \qquad DQ(x,t,-\lambda)\hat{D} = Q(x,t,\lambda),$$
$$D = \mathrm{diag}\,(1,-1,1,-1,1)$$

$$Q_1(x,t) = u_1 E_{e_1-e_2} + u_2 E_{e_2} + u_3 E_{e_1+e_2} + v_1 E_{-e_1+e_2} + v_2 E_{-e_2} + v_3 E_{-e_1-e_2}$$
$$= \begin{pmatrix} 0 & u_1 & 0 & u_3 & 0 \\ v_1 & 0 & u_2 & 0 & u_3 \\ 0 & v_2 & 0 & u_2 & 0 \\ v_3 & 0 & v_2 & 0 & u_1 \\ 0 & v_3 & 0 & v_1 & 0 \end{pmatrix},$$
$$Q_2(x,t) = u_4 E_{e_1} + v_4 E_{-e_1} + w_1 H_{e_1} + w_2 H_{e_2}$$
$$= \begin{pmatrix} w_1 & 0 & u_4 & 0 & 0 \\ 0 & w_2 & 0 & 0 & 0 \\ w_4 & 0 & 0 & 0 & u_4 \\ 0 & 0 & 0 & -w_2 & 0 \\ 0 & 0 & -v_4 & 0 & -w_1 \end{pmatrix},$$
$$J = a_1 H_{e_1} + a_2 H_{e_2} = \mathrm{diag}\,(a_1, a_2, 0, -a_2, -a_1),$$
$$K = b_1 H_{e_1} + b_2 H_{e_2} = \mathrm{diag}\,(b_1, b_2, 0, -b_2, -b_1),$$

Combining both reductions for the matrix elements of $Q_j(x,t)$ we have:

$$v_1 = \epsilon u_1^*, \qquad v_2 = \epsilon u_2^*, \qquad v_3 = \epsilon u_3^*, \qquad v_4 = u_4^*,$$

The commutativity condition for the Lax pair

$$i\left(\frac{\partial V_2}{\partial x} + \lambda\frac{\partial V_1}{\partial x}\right) - i\left(\frac{\partial U_2}{\partial t} + \lambda\frac{\partial U_1}{\partial t}\right) + [U_2 + \lambda U_1 - \lambda^2 J, V_2 + \lambda V_1 - \lambda^2 K] = 0$$

must hold identically with respect to λ. The terms proportional to λ^4, λ^3 and λ^2 vanish identically. The term proportional to λ and the λ-independent term vanish provided Q_i satisfy the NLEE:

$$\begin{aligned}
&i\frac{\partial V_1}{\partial x} - i\frac{\partial U_1}{\partial t} + [U_2, V_1] + [U_1, V_1] = 0,\\
&i\frac{\partial V_2}{\partial x} - i\frac{\partial U_2}{\partial t} + [U_2, V_2] = 0.
\end{aligned}$$

In components the corresponding NLEE:

$$
\begin{aligned}
&-2i(a_1-a_2)\frac{\partial u_1}{\partial t}+2i(b_1-b_2)\frac{\partial u_1}{\partial x}+\kappa\epsilon u_2^*(\epsilon u_2^* u_3-u_1u_2-2u_4)=0,\\
&-2ia_2\frac{\partial u_2}{\partial t}+2ib_2\frac{\partial u_2}{\partial x}-\kappa(u_2\epsilon(|u_3|^2-|u_1|^2)+2u_3u_4^*+2\epsilon u_1^*u_4)=0,\\
&-2i(a_1+a_2)\frac{\partial u_3}{\partial t}+2i(b_1+b_2)\frac{\partial u_3}{\partial x}+\kappa u_2(\epsilon u_2^* u_3-u_1u_2+2u_4)=0,\\
&-2ia_1\frac{\partial u_4}{\partial t}+2ib_1\frac{\partial u_4}{\partial x}+i\frac{\partial}{\partial t}(-(2a_2-a_1)u_1u_2+(2a_2+a_1)\epsilon u_2^*u_3)\\
&+i(2b_2-b_1)\frac{\partial(u_1u_2)}{\partial x}-i(2b_2+b_1)\epsilon\frac{\partial(u_2^*u_3)}{\partial x}-\kappa\left(2\epsilon u_4(|u_1|^2-|u_3|^2)\right.\\
&\left.+\epsilon u_1u_2(|u_1|^2+3|u_3|^2)-u_3u_2^*(3|u_1|^2+|u_3|^2)\right)=0.
\end{aligned}
$$

Let us now introduce

$$
U_4=u_4-\frac{1}{2a_1}((a_1-a_2)u_1u_2+(a_1+a_2)\epsilon u_3u_2^*).
$$

As a result we get:

$$
\begin{aligned}
&-2i(a_1-a_2)\frac{\partial u_1}{\partial t}+2i(b_1-b_2)\frac{\partial u_1}{\partial x}-\frac{\kappa\epsilon}{a_1}u_2^*(2a_1U_4+\epsilon a_2u_2^*u_3+(2a_1-a_2)u_1u_2)=0,\\
&-2ia_2\frac{\partial u_2}{\partial t}+2ib_2\frac{\partial u_2}{\partial x}-\frac{\kappa\epsilon}{a_1}u_2((2a_1+a_2)|u_3|^2-a_2|u_1|^2)\\
&\qquad-2\kappa(u_3U_4^*+\epsilon u_1^*U_4+u_1^*u_2^*u_3)=0,\\
&-2i(a_1+a_2)\frac{\partial u_3}{\partial t}+2i(b_1+b_2)\frac{\partial u_3}{\partial x}+\frac{\kappa}{a_1}u_2(\epsilon(2a_1+a_2)u_2^*u_3-a_2u_1u_2+2a_1U_4)=0,\\
&-2ia_1\frac{\partial U_4}{\partial t}+2ib_1\frac{\partial U_4}{\partial x}+\frac{i\kappa}{a_1}\frac{\partial u_1u_2}{\partial x}-\frac{i\kappa\epsilon}{a_1}\frac{\partial u_2^*u_3}{\partial x}\\
&-\frac{\kappa}{a_1}\left(2\epsilon U_4(|u_1|^2-|u_3|^2)+(\epsilon u_1u_2-u_3u_2^*)((2a_1-a_2)|u_1|^2+(2a_1+a_2)|u_3|^2)\right)=0,
\end{aligned}
$$

One parameter family of MKdV and $so(8)$

Normally with each simple Lie algebra one can associate just one MKdV eq.

The only exception is $so(8)$ which allows a one-parameter family of MKdV equations. The reason is that only $so(8)$ has 3 as a double exponent!

$$\partial_t q_1 = \frac{\partial}{\partial x}\Bigg[2a(\partial_x^2 q_1 - \sqrt{3} q_1 \partial_x q_2) - \sqrt{3}\big[(3a+b)q_4\partial_x q_3 + (3a-b)q_3\partial_x q_4\big]$$

$$- 3q_1\left(2aq_2^2 + (a-b)q_3^2 + (a+b)q_4^2\right)\Bigg],$$

$$\partial_t q_2 = \frac{\partial}{\partial x}\left(\sqrt{3}\partial_x - 6q_2\right)\left[aq_1^2 + \frac{a+b}{2}q_3^2 + \frac{a-b}{2}q_4^2\right]$$

$$\partial_t q_3 = \frac{\partial}{\partial x}\Bigg[-(a+b)\partial_x^2 q_3 + \sqrt{3}q_3\partial_x q_2 - \sqrt{3}\big[(3a+b)q_4\partial_x q_1 + 2bq_1\partial_x q_4\big]$$

$$+ 3q_3\left(2aq_4^2 + (a-b)q_1^2 + (a+b)q_2^2\right)\Bigg],$$

$$\partial_t q_4 = \frac{\partial}{\partial x}\Bigg[-(a-b)(\partial_x^2 q_4 - \sqrt{3}q_4\partial_x q_2) - \sqrt{3}\big[(3a-b)q_3\partial_x q_1) - 2bq_1\partial_x q_3\big]$$

$$+ 3q_4\left(2aq_3^2 + (a-b)q_2^2 + (a+b)q_1^2\right)\Bigg].$$

Hamiltonian properties of the N-wave equations

For standard N-wave equations related to the simple Lie algebra $\mathfrak{g}$:

Introduce grading and basis compatible with it:

$$\tilde{\mathfrak{g}} = \bigoplus_{k=0}^{h-1} \tilde{\mathfrak{g}}^{[s]} \tag{1}$$

Introduce basis in

$$\tilde{\mathfrak{g}}^{[s]} \equiv \mathfrak{h}, \qquad \tilde{\mathfrak{g}}^{[s]} = \mathrm{l.c.}\{E_\alpha^s, \operatorname{hgt}\alpha = s\}.$$

$$[E_\alpha^s, E_\beta^m] = N_{\alpha,\beta} E_{\alpha+\beta}^{s+m}.$$

$$[H, E_\alpha^m] = \alpha(H) E_\alpha^m.$$

$$[E_\alpha, E_{-\alpha}] = H_\alpha.$$

The dual algebra also has a natural grading:

The phase space for the N-wave eqs. – co-adjoint orbit passing through J

The non-trivial Poisson brackets are:

$$\{u_\alpha(x), u^*_\beta(y)\} = \delta_{\alpha,-\beta}\alpha(J)\delta(x-y). \qquad (2)$$

Generalization to polynomial bundles – Reyman, Kulish and Semenov-Tian-Schanskii (1980)

For the new N-wave equations:

Consider Kac-Moody algebra $\tilde{\mathfrak{g}}$ with elements the grading:

$$U(\lambda) = \sum_s U_s \lambda^s, \qquad V(\lambda) = \sum_s V_s \lambda^s, \qquad U_s, V_s \in \tilde{\mathfrak{g}}^{[s]}.$$

$$\tilde{\mathfrak{g}}^* = \mathop{\oplus}_{k=0}^{h-1} \tilde{\mathfrak{g}}^{[*,s]} \qquad (3)$$

Lax operator contains $\sum_{s=0}^{k} U_s(x)\lambda^{k-s}$ where

$$U_s(x) = \sum_{\alpha, \mathrm{hgt}\ \alpha = s} u_{s,\alpha}(x) E_\alpha,$$

Central extension:

$$\mathcal{E}_\alpha = (E_\alpha, c_p),$$

$$[\mathcal{E}_\alpha, \mathcal{E}_\beta]_p = ([E_\alpha, E_\beta], \omega_p(E_\alpha, E_\beta)),$$

where $\omega_p(X, Y)$ is a co-cycle.

$$\omega_p(X, Y) = \int_{-\infty}^{\infty} \mathrm{Res}\ \lambda^{-p-1} \left\langle X(x, \lambda), \frac{\partial Y}{\partial x}(x, \lambda) \right\rangle .$$

Then the Lax equation acquires explicit Hamiltonian form. Some of the Poisson brackets become:

$$\{u_{s,\alpha}(x), u_{m,\beta}(x)\} = -N_{\alpha,\beta} u_{s+m,\alpha+\beta}(x)\delta(x-y) + c\delta_{s+m,p}\delta_{\alpha,-\beta}\langle E_\alpha, E_\beta\rangle \delta'(x-y). \quad (4)$$

Conclusions and open questions

- More classes of new integrable equations: i) higher rank simple Lie algebras; ii) different types of grading; iii) different power k of the polynomials $U(x,t,\lambda)$ and $V(x,t,\lambda)$ and iv) different reductions of U and V.

- These new NLEE must be Hamiltonian. View the jets $U(x,t,\lambda)$ and $V(x,t,\lambda)$ as elements of co-adjoint orbits of some Kac-Moody algebra.

- Apply Zakharov-Shabat dressing method for constructing their N-soliton solutions and study their interactions.

- Analyze the Hamiltonian properties of the new N-wave equations – work in progress.

- Apply the above methods to twisted Kac-Moody algebras – work in progress

Thank you for your attention!