TURBULENCE IN SUPERFLUIDS:
ideas, experiments, numerics and theory

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ABSTRACT

Turbulence in superfluid helium is unusual and presents a challenge to fluid dynamicists because it consists of two coupled, inter penetrating turbulent fluids: the first is inviscid with quantized vorticity, the second is viscous with continuous vorticity. Despite this double nature, the observed spectra of the superfluid turbulent velocity at sufficiently large length scales are similar to those of ordinary turbulence.

After brief historical overview I will present experimental, numerical and theoretical results which explain these similarities, and illustrate the limits of our present understanding of superfluid turbulence.

0.1 Superfluids: experiments and theory

Heike Kamerlingh-Onnes

Nobel prize 1913

"for his investigations on the properties of matter at low temperatures which led, inter alia, to the production of liquid helium". K-O discovered in **1911** superconductivity.

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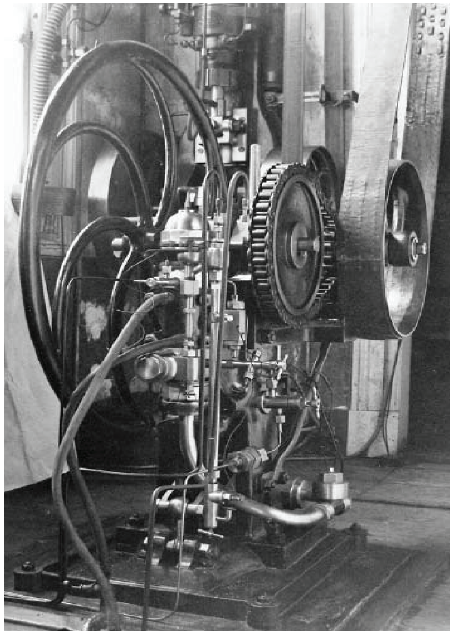
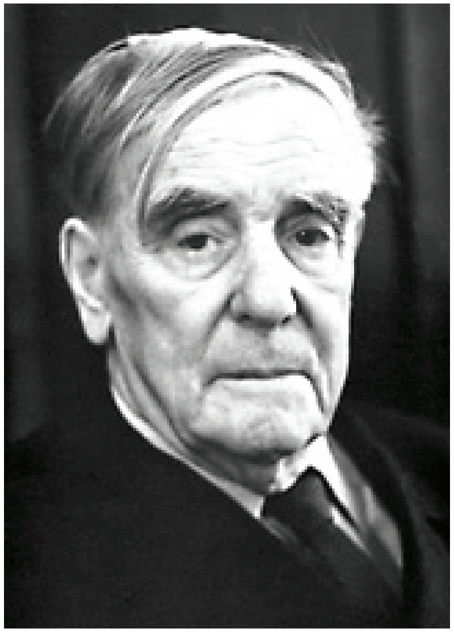


liquified He at $T = 4.2$ K in July 10, **1908**. K-O & coworkers in **1924** discovered density change at $T = 2.18$ K.

Keesom & Wolfke, **1928**: this is a phase transition He I ⇔ He II.

Piotr Leonidovich Kapitza



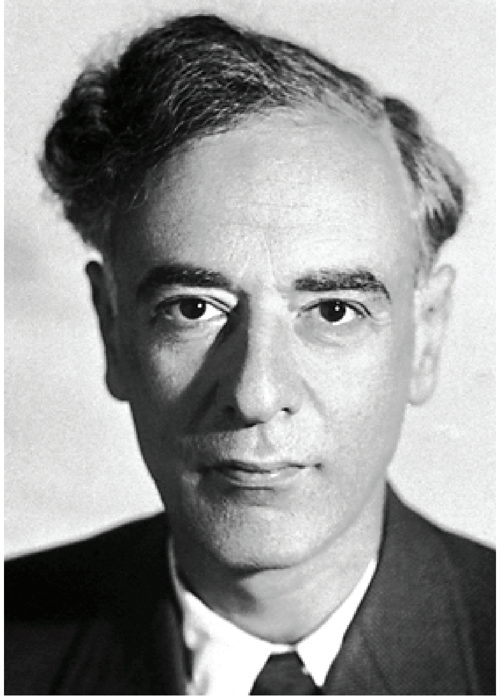
Nobel prize 1978

"for his basic inventions and discoveries in the area of low-temperature physics". P.L. Kapitza in Moscow discovered and named in **1937** superfluidity of ^{4}He

Jack Allen

and his student Donald Missener

independently discovered superfluidity in PLK's Cambridge lab.

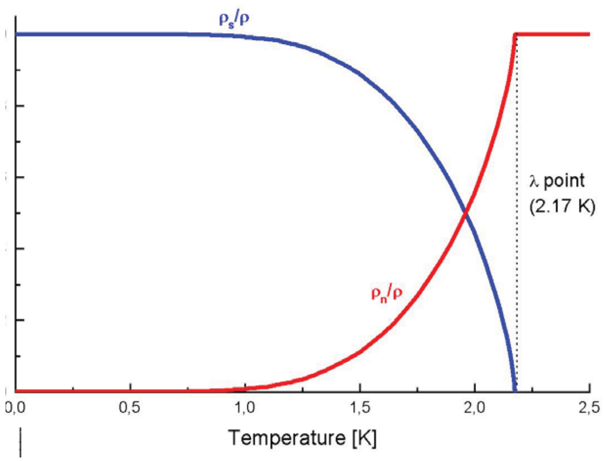


Lev Davidovich Landau
Nobel Prize, 1962
"for his pioneering theories for condensed matter, especially liquid helium".
In particular, he quantized in 1941 the Tisza-1940 two-fluid model and suggested Andronikashvilii's 1946 experiment on oscillating in He II discs.

Elepter Luarsabovich Andronikashvili

Laszlo Tizsa

Its period and damping measures densities of superfluid, $\boldsymbol{\rho}_s$ and normal, $\boldsymbol{\rho}_s$, components:



Landau-Tizsa two fluid model

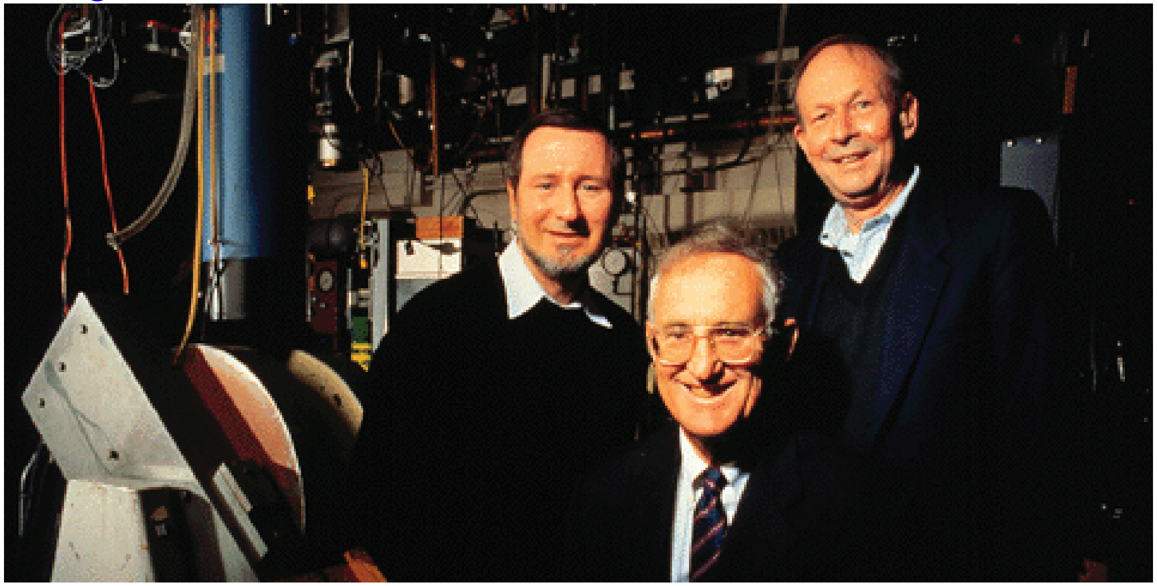
for superfluid, $\boldsymbol{V}_n$, and normal $\boldsymbol{V}_s$ velocities:

$$\rho_s \frac{D\mathbf{v}_s}{Dt} = -\frac{\rho_s}{\rho}\nabla p + \rho_s S \nabla T - \mathbf{F}_{ns}$$

$$\rho_n \frac{D\mathbf{v}_n}{Dt} = -\frac{\rho_n}{\rho}\nabla p - \rho_s S \nabla T + \eta \nabla^2 \mathbf{v}_n + \mathbf{F}_{ns}$$

predicts "second sound", critical velocity, etc.
Here: S – entropy, T – temperature and $\boldsymbol{F}_{ns} = A\rho_n\rho_s(\boldsymbol{V}_s - \boldsymbol{V}_n)^3$ is the mutual friction between superfluid and normal components

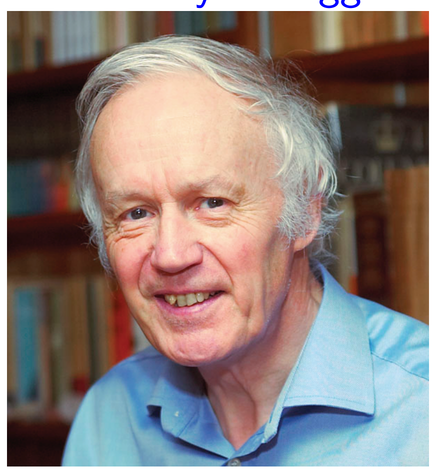
Douglas D. Osheroff, David M. Lee & Robert C. Richardson



Nobel Prize 1996 "for their discovery of superfluidity in helium-3".

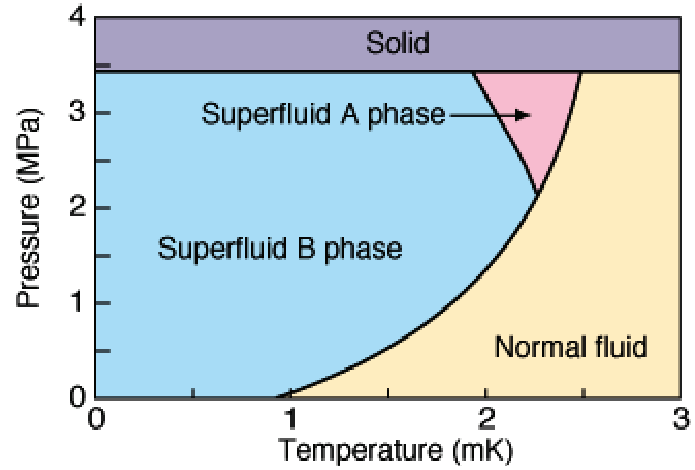
Alexei A. Abrikosov, Vitaly L. Ginzburg, & Anthony J. Leggett





Nobel Prize 2003 "for pioneering contributions to the theory of superconductors and superfluids"

^{3}He, the result of tririum decay, was produced (150 Kg since 1955) and liquified in LANL. Using Pomeranchuk's compressive cooling D.O, R.R&D.L discovered superfluidity of ^{3}He on April 20, 1972 at Cornell.



Knowing this before publication, J. Leggett on Sept. 5, 1972 submited to PRL explanation of their observations as Bardeen-Cooper-Schrieffer condensation of Couper pairs of ^{3}He atoms in the triplet state with the tensorial ordering parameter. The B-state has an isotropic gap.

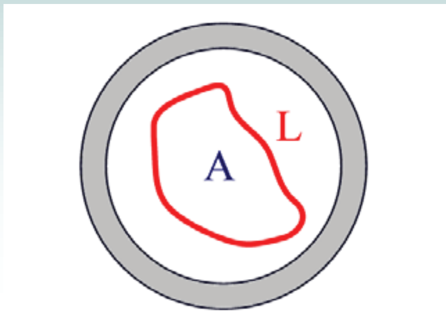
Quantum mechanical description of He II

Macroscopic wave function $\Psi = \sqrt{\rho_s}\exp\{i\varphi(r,t)\}$

$$\hat{p} = i\hbar\nabla \longrightarrow \mathbf{v}_s = \frac{\hbar}{m_4}\nabla\varphi \longrightarrow curl\,\mathbf{v}_s = 0$$

Circulation –singly connected region

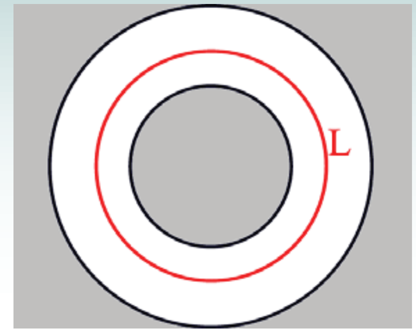
$$\Gamma = \oint_L \mathbf{v}_s d\ell = 0$$



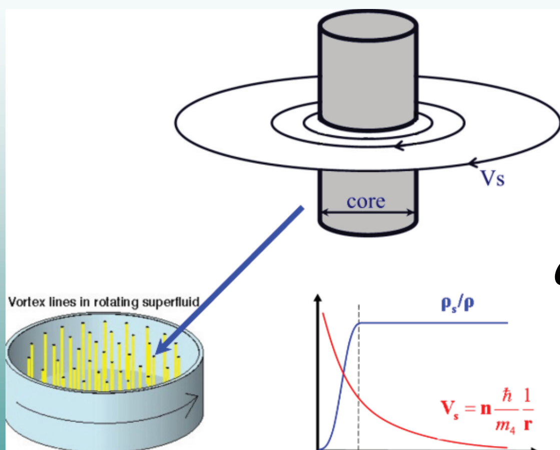
Circulation- multiply connected region

$$\Gamma = \oint_L \mathbf{v}_s d\ell = n\frac{h}{m_4} = n\kappa$$

$$\kappa \cong 10^{-7} m^2 / s$$



Quantized vortices in He II

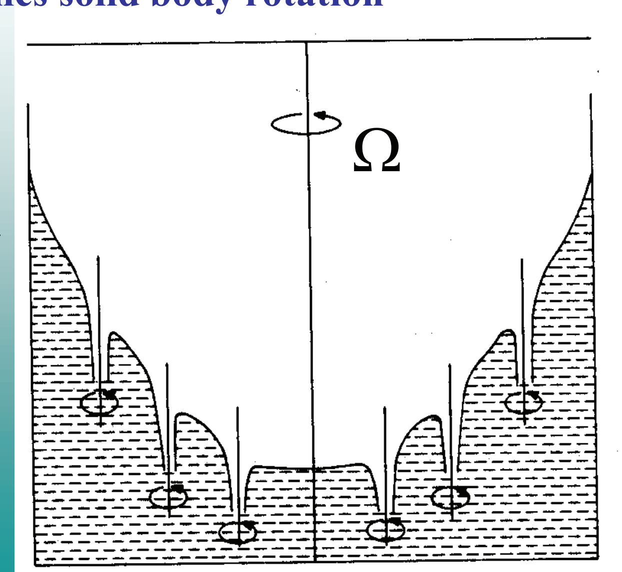


vorticity

$$\omega_N = 2\Omega \cong \langle \omega_S \rangle \cong \kappa L$$

Rotating bucket of He II

-thanks to the existence of rectilinear vortex lines He II mimics solid body rotation

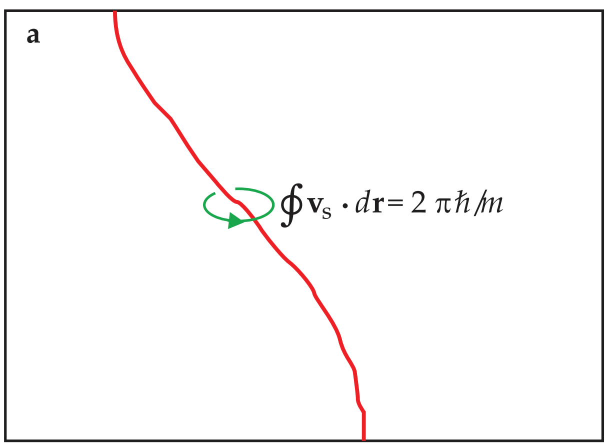


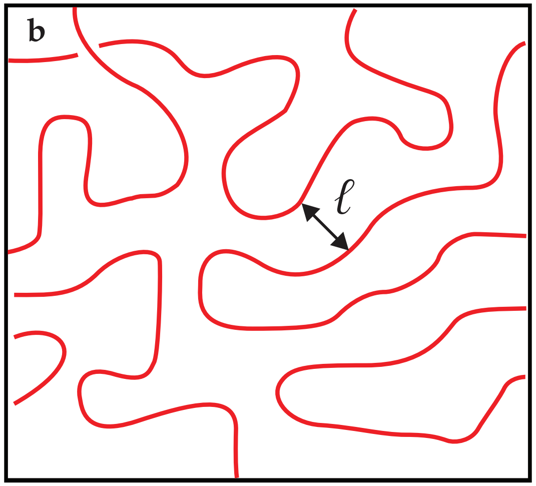
Turbulence in a superfluid was predicted first by Richard Feynman in **1955** and found experimentally (in counterflow ^{4}He) by Henry Hall and Joe Vinen in **1956**. Consider

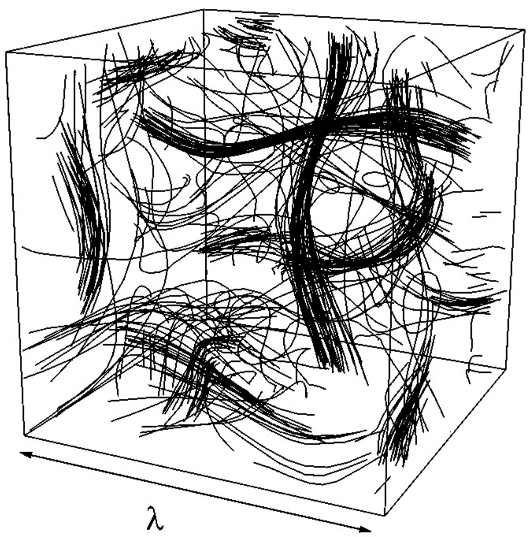
1.3.1 Normal fluid *vs.* superfluid at $T \to 0$ limit:

- Normal fluid kinematic viscosity $\nu \neq 0$ *vs.* $\nu \equiv 0$ in superfluids;
- Two scales in normal fluids: Outer scale $\mathcal{L}$ and dissipative micro-scale $\eta \ll \mathcal{L}$;
- Two additional scales in superfluids due to quantization of vortex lines:

$\Downarrow$ vortex core diameter a_0 $\Downarrow$ $\quad$ $\Downarrow$ mean inter-vortex distance ℓ $\Downarrow$ $\quad$ $\Downarrow$ Outer scale $\mathcal{L}$ $\Downarrow$

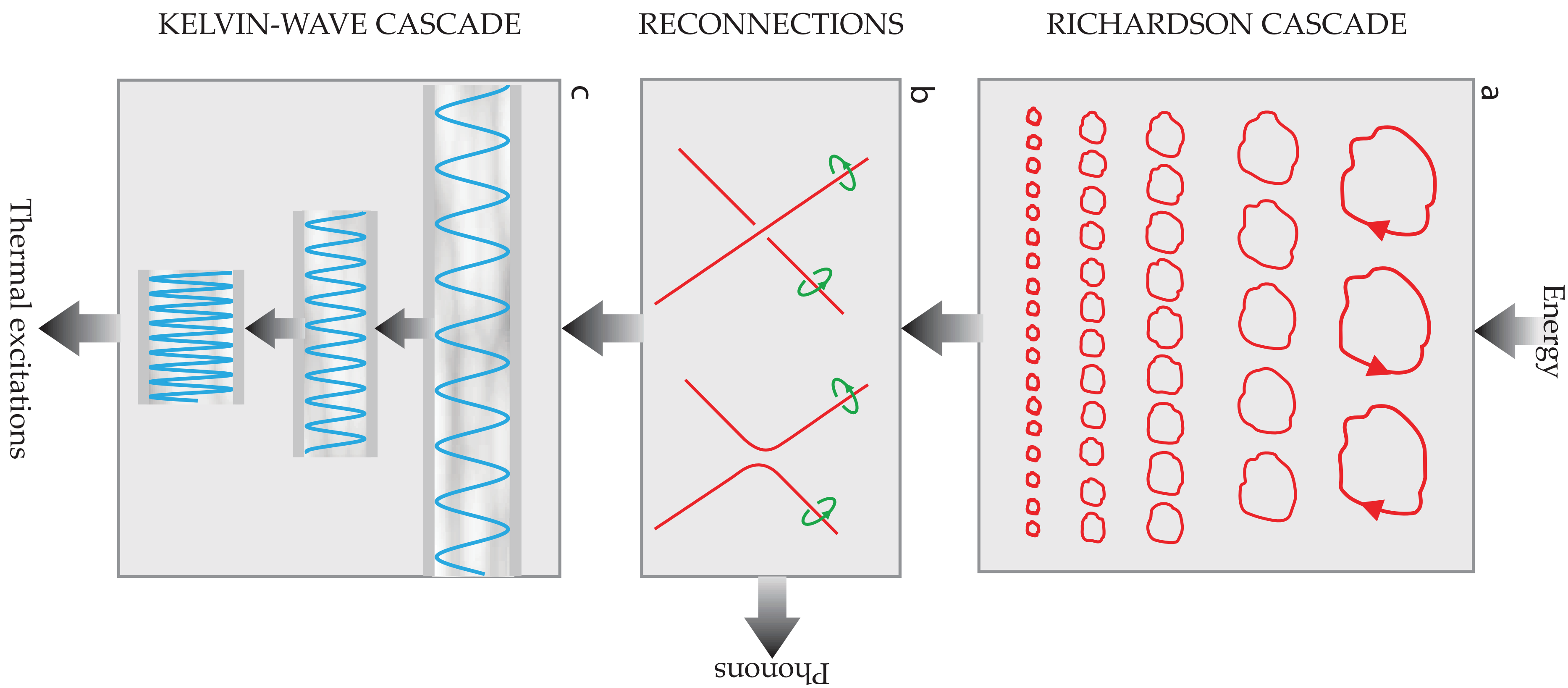






In ^{4}He $a_0 \simeq 1$ Å, in ^{3}He $a_0 \simeq 800$ Å. Experimentally, in both ^{4}He and ^{3}He, $\Lambda \equiv \ln\left(\frac{\ell}{a_0}\right) \simeq 12 \div 15$

Sketch of the quantum-turbulence cascades[1]



Very small scales:

$L \ll \ell.$

Intervortex scales:

$L \sim \ell$

Large scales:

$L \gg \ell$

Physics of different temperature regions in superfluid ^{4}He

- Near T_λ, phase transition region, $2.16\,\text{K} < T < T_\lambda$. Will not be discussed today.
  small superfluid density: large thermodynamical fluctuation, large mutual friction $\alpha \simeq 1$.
  Very interesting region to study interplay of phase-transition with superfluid turbulence phenomena.

- Medium temperatures, two-fluid region, $1.5\,\text{K} \lesssim T \lesssim 2.16\,\text{K}$. Will be discussed shortly.
  ρ_s & ρ_n are comparable, $\left(\frac{\rho_s}{\rho_n}\right.$ between 10 and 0.1$\left.\right)$, large mutual friction $0.1 \lesssim \alpha \lesssim 1$.
  "Quasi-classical" behavior, with interesting new physics of superfluid turbulence: unexpected spectra of vortex-line density (*requires more detailed theoretical analysis*), encasement of intermittency (*requires further numerical and experimental study*), bottleneck energy accumulation near intervortex scale (*much more analytical, numerical and experimental works are needed*).

- Low temperatures, $0.8\,\text{K} \lesssim T \lesssim 1.3\,\text{K}$, transient from two- to one-fluid region. Will be discussed.
  $\rho_n < 0.05\rho$, small, but mutual friction $0.0007 < \alpha < 0.04$.
  I expect classical intermittency exponents. Kelvin waves (KWs) still effectively are damped. Both predictions require experimental and further numerical clarification.

- Ultra-low temperatures, $T \lesssim 0.6\,\text{K}$, one-fluid hydrodynamic and KW region. Main subject today.
  Statistical importance of KW cascade. L'vov-Nazarenko spectrum of ***weak*** turbulence of KWs vs. Vinen spectrum of ***strong*** KW turbulence – the problem in its infancy.

- Zero-temperature limit $T \lesssim 0.06\,\text{K}$. There are no reason to discuss separately from $T \lesssim 0.6\,\text{K}$.
  Mutual friction fully irrelevant, KW damping is assumed to be caused by photon emission and KW cascade probably reaches the core radius – Interesting, important and difficult problem to study.

Two-fluid hydrodynamic (HD) equations for medium and low temperatures, $T \gtrsim 0.8$ K

In the HD region, $R \gg \ell$, one can neglect the quantization of vortex lines and make use coarse-grained two fluid equation for velocities the superfluid and normal components $\boldsymbol{u}_{\rm s}$ and $\boldsymbol{u}_{\rm n}$, with densities $\rho_{\rm s}$ and $\rho_{\rm n}$ and pressures $p_{\rm s}$ and $p_{\rm n}$

$$\rho_{\rm s}\Big[\frac{\partial\,\boldsymbol{u}_{\rm s}}{\partial t}+(\boldsymbol{u}_{\rm s}\boldsymbol{\nabla})\boldsymbol{u}_{\rm s}\Big]-\boldsymbol{\nabla}p_{\rm s} \;=\; -\boldsymbol{F}_{\rm ns}\,,\quad p_{\rm s}=\frac{\rho_{\rm s}}{\rho}[p-\rho_{\rm n}|\boldsymbol{u}_{\rm s}-\boldsymbol{u}_{\rm n}|^2]\,, \tag{1a}$$

$$\rho_{\rm n}\Big[\frac{\partial\,\boldsymbol{u}_{\rm n}}{\partial t}+(\boldsymbol{u}_{\rm n}\boldsymbol{\nabla})\boldsymbol{u}_{\rm n}\Big]-\boldsymbol{\nabla}p_{\rm n} \;=\; \rho_{\rm n}\nu\Delta\boldsymbol{u}_{\rm n}+\boldsymbol{F}_{\rm ns}\,,\quad p_{\rm n}=\frac{\rho_{\rm n}}{\rho}[p+\rho_{\rm s}|\boldsymbol{u}_{\rm s}-\boldsymbol{u}_{\rm n}|^2]\,, \tag{1b}$$

coupled by the the mutual friction between superfluid and normal components of the liquid mediated by quantized vortices which transfer momenta from the superfluid to the normal subsystem and vice versa:

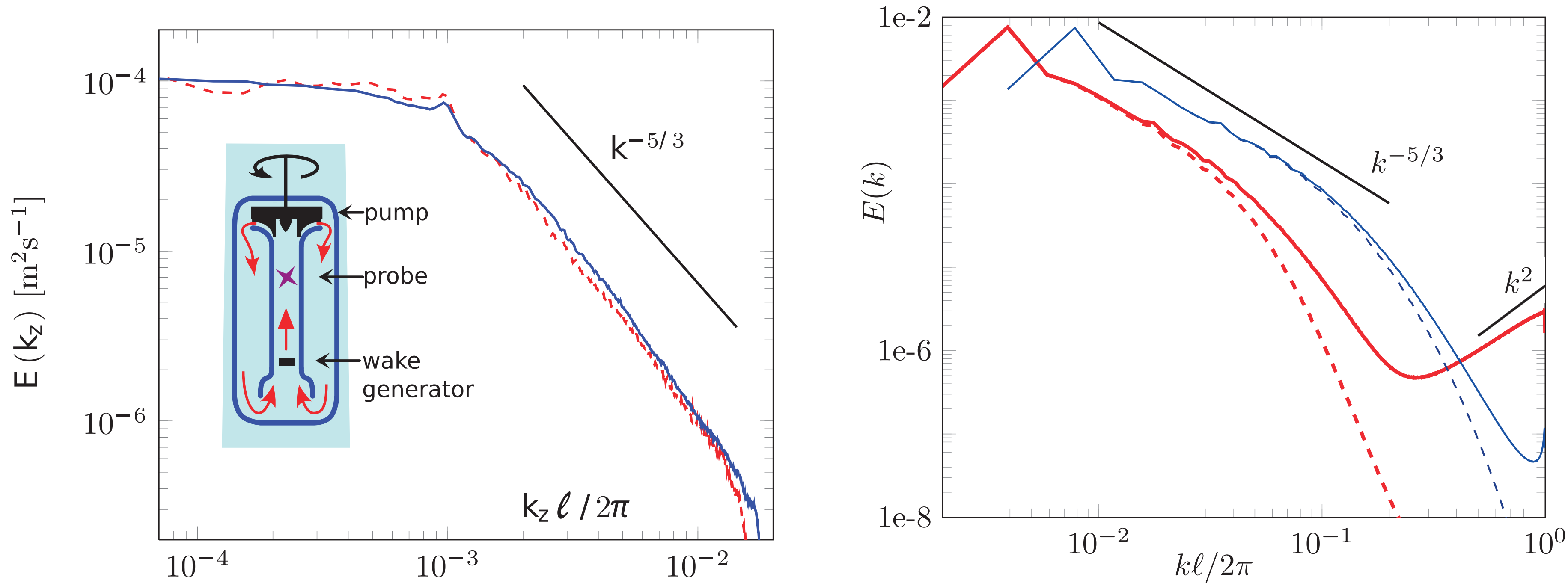
$$\boldsymbol{F}_{\rm ns}=-\rho_{\rm s}\{\alpha'(\boldsymbol{u}_{\rm s}-\boldsymbol{u}_{\rm n})\times\boldsymbol{\omega}_{\rm s}+\alpha\,\hat{\boldsymbol{\omega}}_{\rm s}\times[\boldsymbol{\omega}_{\rm s}\times(\boldsymbol{u}_{\rm s}-\boldsymbol{u}_{\rm n})]\}\approx\alpha\,\rho_{\rm s}\omega_{_{\rm T}}(\boldsymbol{u}_{\rm s}-\boldsymbol{u}_{\rm n})\,,\quad \omega_{_{\rm T}}=\sqrt{\langle|\boldsymbol{\omega}_{\rm s}|^2\rangle}\,. \tag{1c}$$

Eqs (1) are very similar to the Navier-Stokes equation. Therefore in a theory of large-scale superfluid turbulence we can use numerous tools, developed in the theory of classical HD turbulence, in particular, the differential closure for the energy flux

$$\boldsymbol{\varepsilon(k)}=-\frac{1}{8}\sqrt{\boldsymbol{k}^{11}\,\boldsymbol{E(k)}}\,\frac{\boldsymbol{d}}{\boldsymbol{dk}}\Big[\frac{\boldsymbol{E(k)}}{\boldsymbol{k}^2}\Big]\qquad\Rightarrow\qquad \boldsymbol{E(k)}=\boldsymbol{k}^2\Big[\frac{\boldsymbol{24\,\varepsilon}}{\boldsymbol{11k^{11/2}}}+\Big(\frac{\boldsymbol{T}}{\boldsymbol{\pi\rho}}\Big)^{3/2}\Big]^{2/3}. \tag{2}$$

This solution with the constant energy flux $\varepsilon(k)=\varepsilon$ gives KO-41 spectrum $\propto k^{-5/3}$ for small k end thermodynamic equilibrium spectrum $T/\pi\rho$ at large k.

0.3 Kolmogorov spectra in ^{4}He turbulence



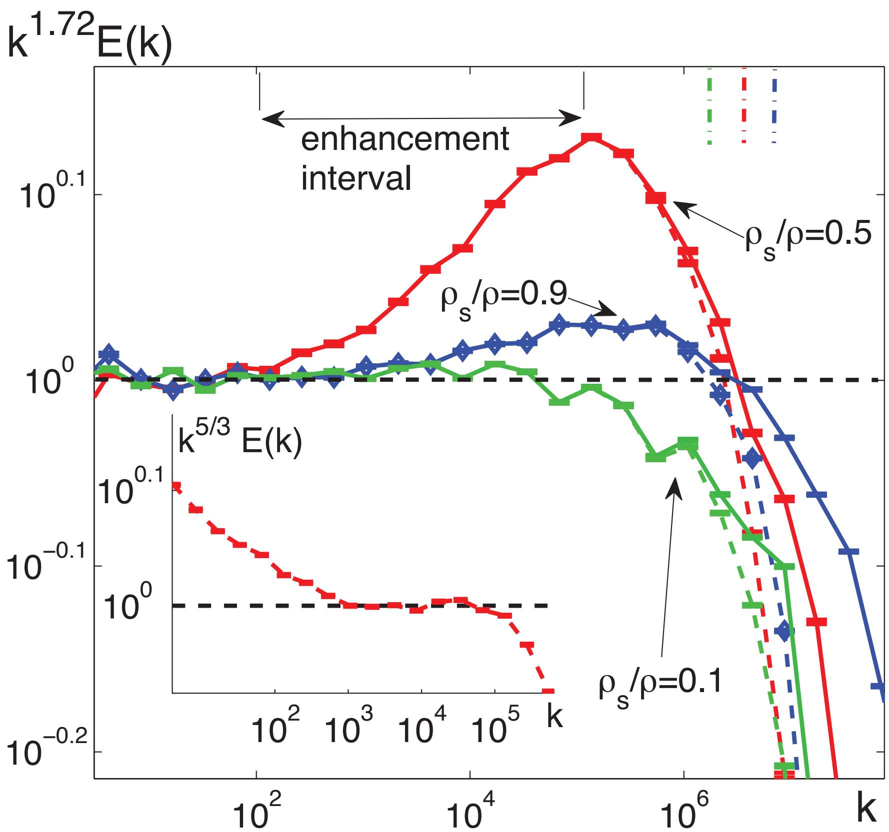
Energy spectrum measured in the TOUPIE wind tunnel (Inset) below the superfluid transition (solid blue line, $T = 1.56$ K and aboveT_λ (dashed red line) [1]
Right: Numerical energy spectra of the superfluid (solid lines) and normal (dashed lines) component from two-fluid Eqs. at $T = 1.15$ K (red) and $T = 2.157$ K (blue) with truncation of phase space beyond the intervortex scale [2]

[1] Salort J, Chabaud B, Lvque E, Roche P-E, Energy cascade and the four-fifths law in superfluid turbulence. Europhys Lett. 97, 34006 (2012)
[2] C. F. Barenghi, V. S. Lvov, and P.-E. Roche, Experimental, numerical, and analytical velocity spectra in turbulent quantum fluid, Proc Natl Acad Sci USA., 111 46834690 (2014)

0.4 Intermittency enhancement in ^{4}He turbulence

L. Boue, V.S. L'vov, A. Pomyalov, I. Procaccia, PRL, 110, 014502 (2013)

Our shell model simulations with eight decades of k-space allowed detailed comparison of classical and superfluid turbulent statistics in the wide temperature range. A difference between classical and superfluid intermittent behavior in a wide (up to three decades) interval of scales was found in the range $0.8T_\lambda < T < 0.9T_\lambda$, where ($\rho_s \approx \rho_n$)



Superfluid (solid lines) and normal fluid (dash lines) compensated energy spectra $k^{1.72}E(k)$; the compensation factor is the classical energy spectrum with intermittency correction.

Inset: $k^{5/3}E(k)$ for $T = 0.9\,T_\lambda$. Shell model simulation of Eqs. (1) at $T/T_\lambda = 0.99$ K(green), 0.9(red) and 0.85(blue), corresponding to $\rho_s/\rho = 0.1\,,\ 0.5\,,$ and 0.9 respectively. The vertical dash lines indicate $k_\ell \equiv 1/\ell$.

1 Weak turbulence of small-scale Kelvin waves in zero temperature limit

The theory is based on Biot-Savart equation for quantized vortex lines, written in the Hamiltonian form[3], [4] for small amplitudes $a_k(t)$, of Kelvin waves (KWs) that describe deviations of the vortex line from the straight line. Using general approach, described, e.g. in ZLF-92 book [5] the effective KW Hamiltonian was found in LLNR-10 [6].

$$\mathcal{H}_{\text{eff}} = \sum_k \omega(k) b_k b_k^* + \frac{1}{36} \sum_{1+2+3=4+5+6} \widetilde{W}_{1,2,3}^{4,5,6}\, b_1 b_2 b_3 b_4^* b_5^* b_6^* \,. \tag{3a}$$

$$\widetilde{\mathcal{W}}_{k,1,2}^{3,4,5} = W_{k,1,2}^{3,4,5} + Q_{k,1,2}^{3,4,5} \simeq -\frac{3kk_1k_2k_3k_4k_5}{4\pi\kappa} \,. \tag{3b}$$

Next step is to derive the $3 \leftrightarrow 3$-KW Kinetic Equation (KE) for the "occupation numbers" $n(k,t)$ [7]:

$$\frac{\partial n(k,t)}{\partial t} = \frac{\pi}{12} \int\!\!\!\int\!\!\!\int\!\!\!\int\!\!\!\int \left| \widetilde{\mathcal{W}}_{k,1,2}^{3,4,5} \right|^2 \delta_{k,1,2}^{3,4,5}\, \delta\left({}^{\Lambda}\Omega_{k,1,2}^{3,4,5} \right) n_k n_1 n_2 n_3 n_4 n_5$$
$$\times \left(n_k^{-1} + n_1^{-1} + n_2^{-1} - n_3^{-1} - n_4^{-1} - n_5^{-1} \right) dk_1\, dk_2\, dk_3\, dk_4\, dk_5 \,. \tag{4}$$

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[3] **E.B. Sonin, Reviews of modern physics v. 59, 87 (1987)**
[4] **B. V. Svistunov, Phys. Rev. B v. 52, 3647 (1995)**
[5] **ZLF-92: V.E. Zakharov, V.S. L'vov & G.E. Falkovich, *Kolmogorov Spectra of Turbulence*, (Springer-Verlag, 1992)**
[6] **LLNR-10: J. Laurie, V. S. Lvov, S. Nazarenko & O. Rudenko, Phys. Rev. B., v. 81, 104526 (2010)**
[7] Here, we evoke a quantum mechanical analogy as an elegant shortcut, allowing us to derive KE and the respective solutions easily. However, the reader should not get confused with this analogy and understand that our KW system is purely classical. In particular, the Plank's constant $\hbar$ is is irrelevant outside of this analogy, and should be simply replaced by 1.

Stationarity of solutions of Eq. (4) was found by Kosik-Svistunov (KS) [8] under assumption of interaction locality [convergence of all integrals in (4)]:

$$\boldsymbol{E}_{\mathrm{KS}}(\boldsymbol{k}) = \boldsymbol{\omega}_k \boldsymbol{n}_k = \boldsymbol{C}_{\mathrm{KS}} \frac{\boldsymbol{\Lambda} \boldsymbol{\kappa}^{7/5} \boldsymbol{\epsilon}^{1/5}}{\boldsymbol{k}^{7/5}}\,, \quad \text{KS-spectrum of weak KW turbulence ?} \tag{5}$$

However, as shown in LLNR-10 paper, this locality assumption is wrong and therefore the KS spectrum (5) is irrelevant. Rigorous analysis by L'vov and Nazarenko (LN) [9] culminate with the result:

$$\boldsymbol{E}(\boldsymbol{k}) = \frac{\boldsymbol{C}_{\mathrm{LN}} \boldsymbol{\Lambda} \boldsymbol{\kappa}\, \boldsymbol{\varepsilon}^{1/3}}{\boldsymbol{\Psi}^{2/3} \boldsymbol{\ell}^{4/3} \boldsymbol{k}^{5/3}}\,, \quad \boldsymbol{C}_{\mathrm{LN}} \simeq \mathbf{3.04}\,, \qquad \text{LN-spectrum of weak KW turbulence !} \tag{6}$$

Earlier Vinen suggested the spectrum [10]

$$\boldsymbol{E}(\boldsymbol{k}) \simeq \frac{\boldsymbol{\kappa}^2}{\boldsymbol{k}}\,, \quad \text{Vinen-spectrum of strong KW turbulence !} \tag{7}$$

There is a vast and constantly growing body of literature, where KWs was numerically detected, [11] [12] [13] [14] [15] [16] [17] [18] [19] however the resolution was not sufficient to distingue between KS and LN spectra.

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[8] E. Kozik & B. Svistunov, Phys. Rev. Lett. v. 92, 035301 (2004)
[9] V. S. L'vov & S. Nazarenko, Pis'ma v ZhETF, v. 91, 464 (2010).
[10] W. F. Vinen & J. J. Niemela,, J. Low Temp. Phys. 128, 167 (2002).
[11] W. F. Vinen, M. Tsubota, and A. Mitani, JPhys. Rev. Lett. 91, 135301 (2003).
[12] C. F. Barenghi, L. Skrbek, and K. R. Sreenivasan, Proc. Nath, Acad. Sci. USA 111 4647-4652 (2014).
[13] R. Hanninen and A.W. Baggaley, Proc. Nath. Acad. Sci. USA 111, 46674674 (2014).
[14] L. Kondaurova, V. Lvov, A. Pomyalov, and I. Procaccia, Phys. Rev. B, 89, 014502 (2014).
[15] E. Kozik and B. Svistunov, Phys. Rev. B 72, 172505 (2005).
[16] T. Araki and M. Tsubota, J. Low Temp. Phys. 121, 405 (2000).
[17] S. K. Nemirovskii, J. Pakleza, and W. Poppe, Russ. J. Eng. Thermophys. 3, 369 (1993).
[18] D. Kivotides, J. C. Vassilicos, D. C. Samuels, and C. F. Barenghi, Phys. Rev. Lett. 86, 3080 (2001).
[19] M. Tsubota, T. Araki, S.K. Nemirovskii, Phys. Rev. B 62, 11751-11762 (2000).

Krstutovic [19] and Baggaley–Laurie [20]: *Lvov-Nazarenko* vs *Kozik-Svistunov* controversy is over

DNS of Gross-Pitaevskii equations in [19][a] allows author to conclude that "Numerical data obtained from long time integration and ensemble average over initial conditions support the spectrum proposed by Lvov-Nazarenko" (with $m = 11/3 \simeq 3.67$) ...and exclude the Kozik-Svistunov prediction ($m = 17/5 = 3.4$) see Table in Ref. [13]:

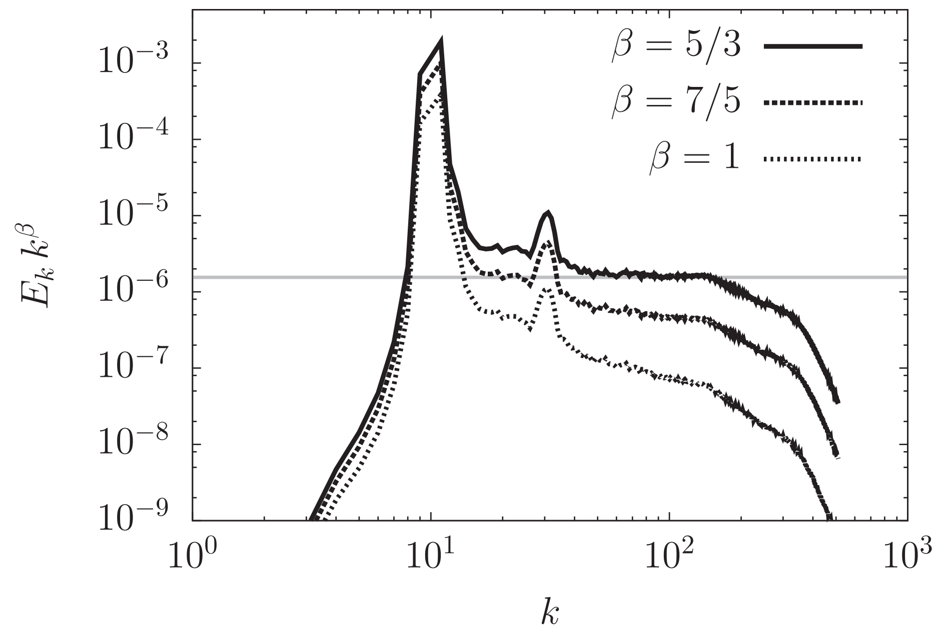
TABLE I. List of runs. $N_\perp$ and N_z are the resolutions in the perpendicular and parallel directions with respect to the vortex. $N_{\rm rea}$ is the number of realizations. n is the number of initial KW modes and m is the exponent k^{-m} of the KW spectrum.

Run	$N_\perp$	N_z	$N_{\rm rea}$	n	ξ	A	m
I	256	128	31	3	0.025	2ξ	3.85 ± 0.24
II	256	128	31	2	0.025	4ξ	3.682 ± 0.13
III	256	256	11	2	0.025	4ξ	3.753 ± 0.17

[a] [19] **G. Krstutovic, PRE 86, 055301(R) (2012)**

DNS of Biot-Savart equation in **20** [a] " observes a remarkable agreement with the Lvov-Nazarenko spectrum" with $C^{\rm num}_{\rm LN} \approx 0.3079$, which agrees with $C^{\rm theory}_{\rm LN} \approx 0.308$, while $C^{\rm num}_{\rm KS} \approx 0.009$ which clearly disagrees with KS-estimate $C^{\rm theory}_{\rm KS} \sim 1$

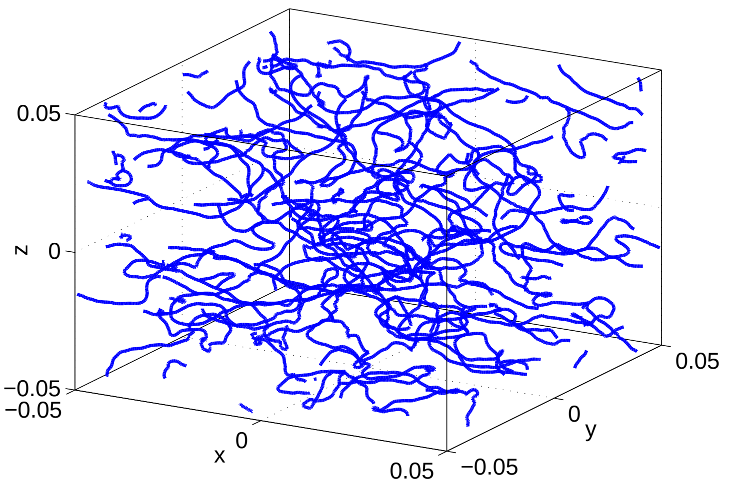
Log-log plots of the DNS energy spectra $E(k)$ compensated by k^β with the LN ($\beta = 5/3$), KS ($\beta = 7/5$) and Vinen ($\beta = 1$) exponents.

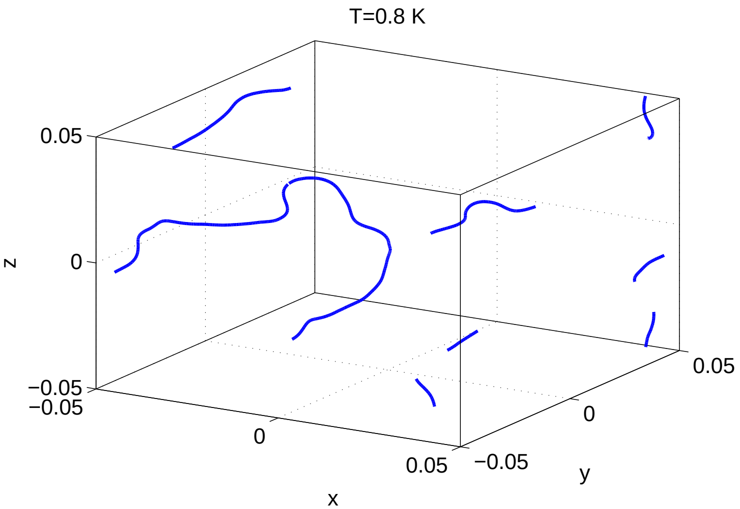


[a] [20] **A. W. Baggaley and J. Laurie, Physical Review B 89, 014504 (2014)**

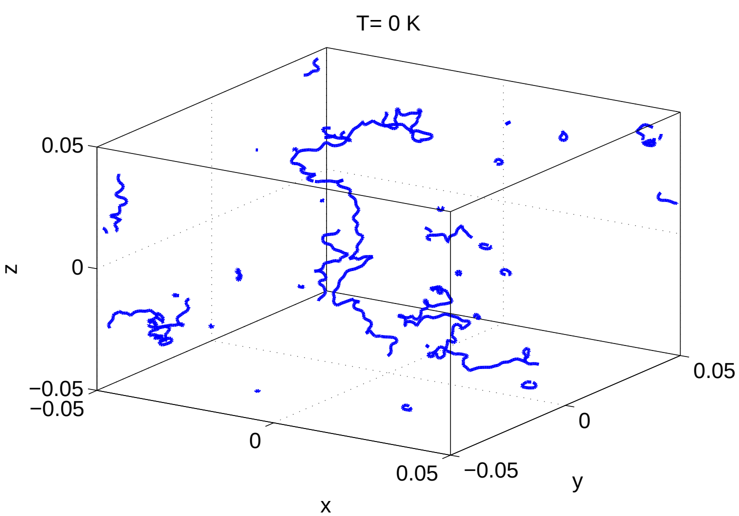
1.1 Kelvin waves and the decay of quantum superfluid turbulence:

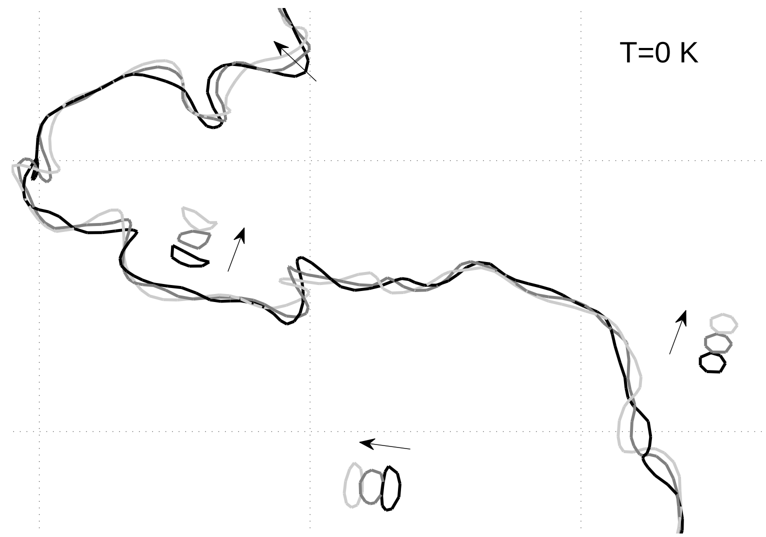
L. Kondaurova, V. Lvov, A. Pomyalov, and I. Procaccia, PRB submitted, (2014)



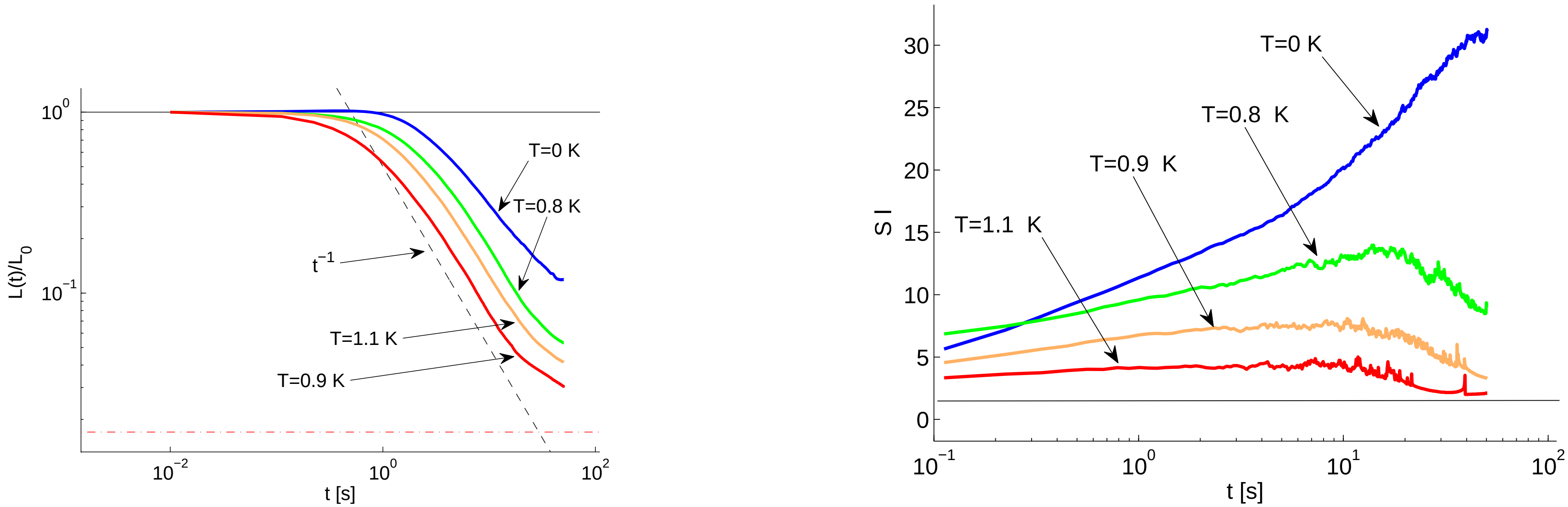


Left: An example of the initial configuration, used in the simulations at all the temperatures.
Right: Configurations obtained at t = 50 s at $T = 0.8$ K.

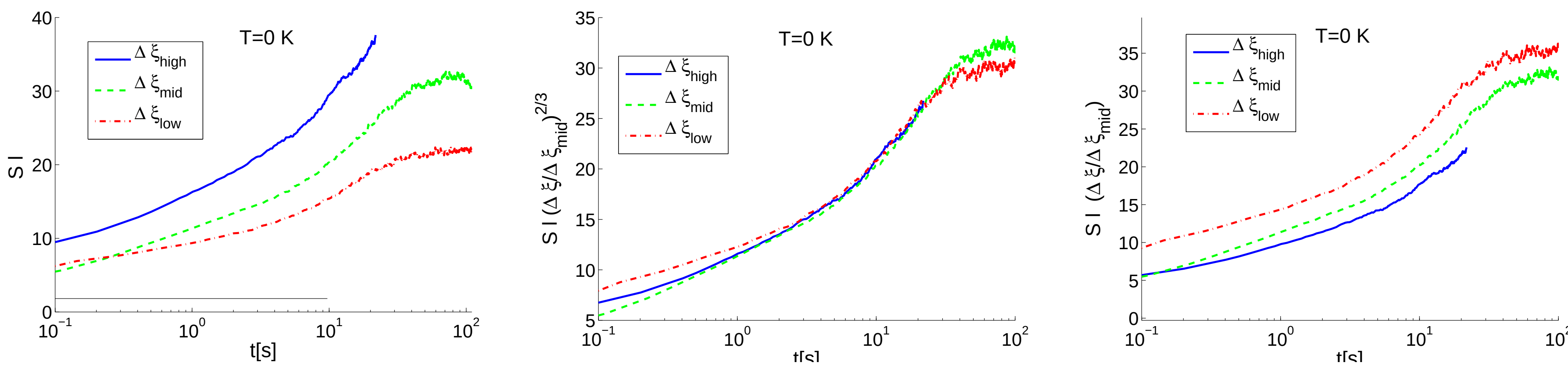




Left: Configurations obtained at t = 50 s at $T = 0$ K.
Right: The fragment of this tangle configuration, shown at three successive times separated by 5 10-4 s. The black lines correspond to the earliest time t = 50 s, the light grey to the latest time. The arrows indicate the direction of the line movement.



Left: The time dependence of normalized VLD $\mathcal{L}(t)/\mathcal{L}(0)$, for different T. The black dashed line shows the asymptotical decay law $\mathcal{L}(t) \propto 1/t$
Right: The evolution of the curvature $S(t)\ell(t)$, normalized by the current intervortex distance $\ell(t)$. Thin horizontal lines show initial curvature



Left: Comparison of the time dependence of the normalized curvature $S(t)\ell(t)$ for different resolutions and $T=0$.
Middle: Normalization by Lvov-Nazarenko spectrum of weak wave turbulence: $\ell S \simeq (\ell\Delta\xi)^{2/3}$ leads to the data collapse
Right: Normalization by Vinen spectrum fo strong wave turbulence $\ell S \simeq (\ell\Delta\xi)^{2/3}$

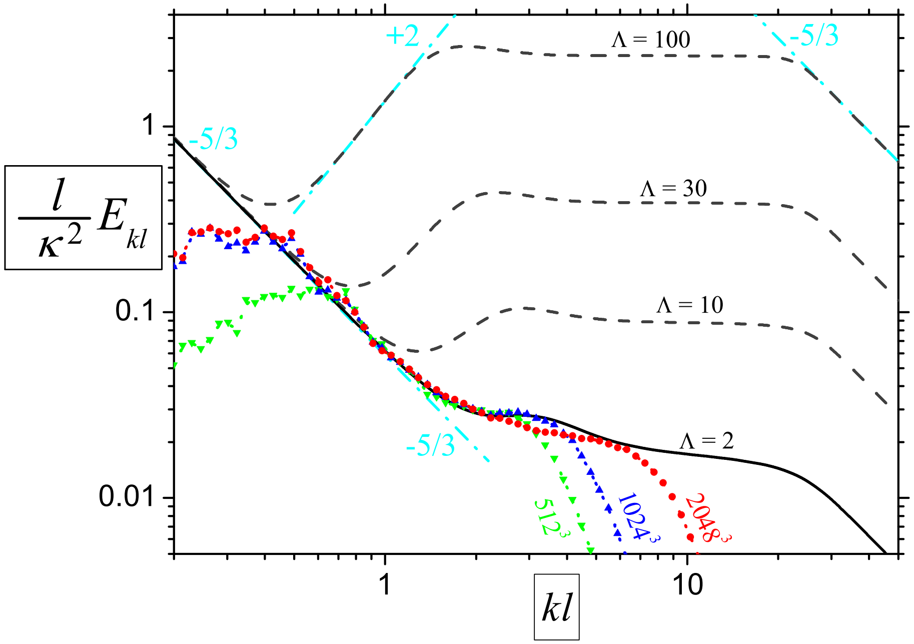
These support WEAK KW turbulence regime (with LN-spectrum) rather then STRONG KW turbulence regime in the vortex tangle decay

2 Bottleneck energy accumulation at cross-over scales at ultralow temperatures and $T \to 0$ limit

2.1 Differential model for small-scale KW turbulence

suggested in [20], approximates superfluid turbulence and KW-motions results for $T = 0$:

$$\varepsilon(k) = -\Big\{\frac{1}{8}\sqrt{k^{11} g(k\ell) E(k)} + \frac{3}{5}\frac{\big\{\Psi k^3 k_* \ell^2 [1 - g(k\ell)] E(k)\big\}^2}{(C\Lambda\,\kappa)^3} \times \frac{d}{dk}\Big\{E(k)\Big[\frac{g(k\ell)}{k^2} + \frac{[1-g(k\ell)]}{k_*^2}\Big]\Big\},$$



I. For $k\ell \ll 1$ $E(k)$ and $\varepsilon(k)$ are dominated by HD components and one sees K41 law (??), $E^{\rm HD}(k) \propto k^{-5/3}$, with constant HD energy flux.

II. At $k\ell \lesssim 1$ and for $\Lambda \gg 1$ one sees the bottleneck with thermodynamic equilibrium: equipartition between HD degrees of freedom, $E^{\rm HD}(k) \propto k^2$.

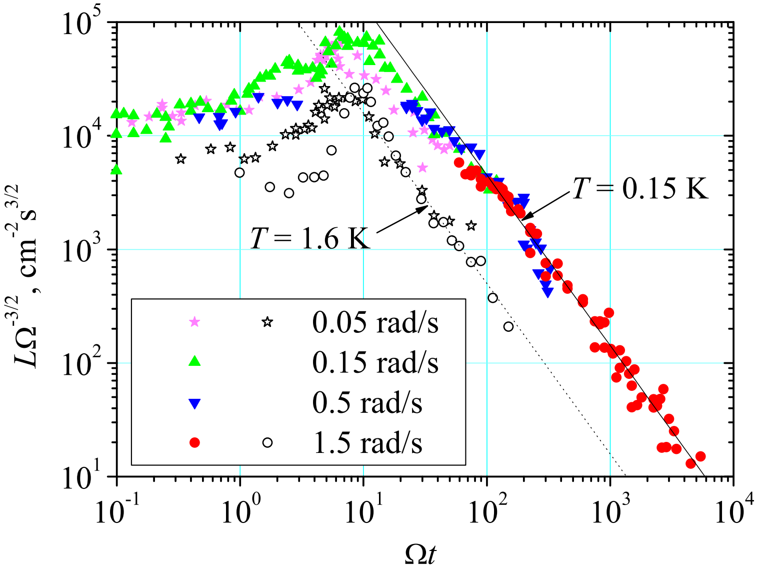
III. At $k\ell \gtrsim 1$ the energy flux is still carried by HD motions, $\varepsilon(k) \simeq \varepsilon^{\rm HD}(k)$ while energy is already dominated by KWs, $E(k) \simeq E^{\rm KW}(k)$. In the flux-free system of KWs one again sees thermodynamic equilibrium: but with equipartition between KW degrees of freedom, $E^{\rm KW}(k)$ =const

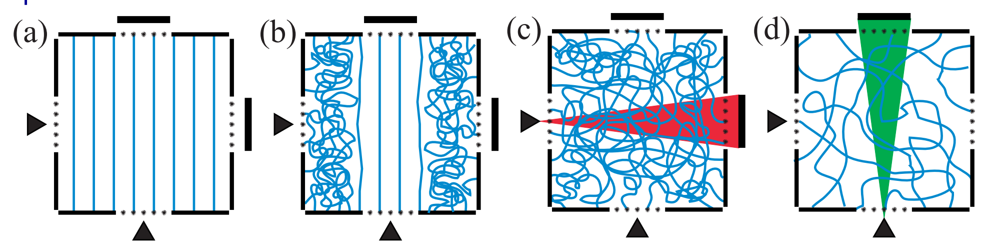
IV. For $k\ell \gg 1$ $E(k) \simeq E^{\rm KW}(k)$ and $\varepsilon(k) \simeq \varepsilon^{\rm KW}(k)$ i.e. are dominated by KWs. In this pure KW region, as expected, one observes the LN-spectrum of KWs, $E^{\rm KW}(k) \propto k^{-5/3}$ with constant KW-energy flux.

• As Λ decreases, the bottleneck effect becomes less pronounced. The equilibrium HD region II practically disappears for $\Lambda \simeq 2$. However the equilibrium KW region III is still well featured, being less sensitive to Λ. This agrees with the Tokio-DNS results [21] for 2048^3, 1024^3, and 512^3, shown by dots.

[20] V. S. L'vov, S. V. Nazarenko & O. Rudenko, Phys. Rev. B 76, 024520 (2007).
V. S. L'vov, S. V. Nazarenko & O. Rudenko, J. of Low Temp. Phys. v. 153, 140 (2008).
[21] N. Sasa, M. Machida, T. Kano, V. S. L'vov, O. Rudenko and M. Tsubota,, Phys. Rev. B, 84, 054525 (2011).

– Comparison with the Manchester ^{4}He spin-down [22] and towed grid experiments





⇑ Cartoon of the vortex configurations ⇑

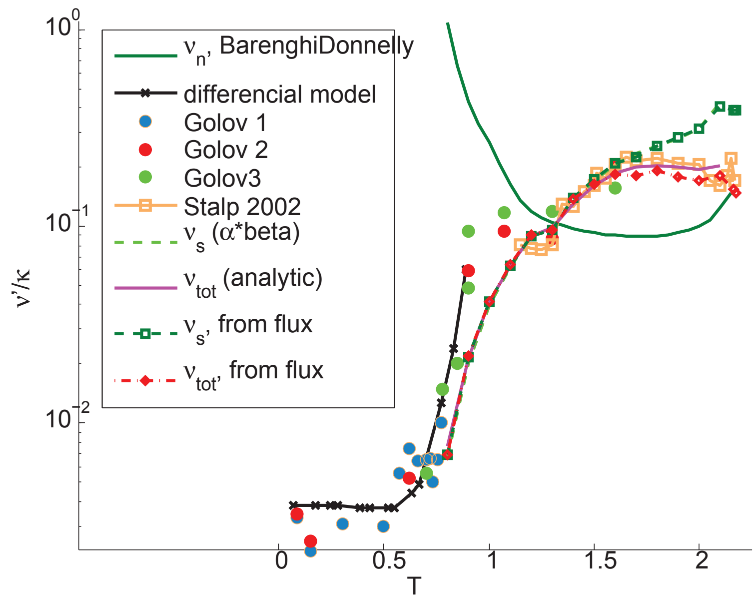
⇐ Vortex line density $(L\,\Omega^{-3/2})$ $vs.$ $(\Omega\, t)$.

Measuring the time-decay of the vortex line density by negative-ion scattering, they found the temperature dependence of the effective viscosity ν', defined via rate of energy dissipation ε and mean square vorticity:

$$\frac{dE(t)}{dt} = \varepsilon(t) = \nu' \langle|\omega|^2\rangle\ ,\ \langle|\omega|^2\rangle = (\kappa L)^2\,,$$

$$\text{Turb. Energy } E \propto \varepsilon^{2/3} \Rightarrow E(t) \propto (t - t_*)^2$$

$$\Rightarrow L(t) \propto 1/[\kappa\sqrt{\nu'(t-t_*)^3}]$$



[22] **Walmsley, Golov, Hall, Levchenko and Vinen**, **PRL, v. 99, 265302 (2007)**

3 Summary and perspectives

Much more experimental, analytical and numerical studies are required to achieve desired level of understanding of superfluid turbulence