Nonlinear waves in two-component Bose-Einstein condensates

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In honor of V.E. Zakharov 75th birthday

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This work was done in collaboration with

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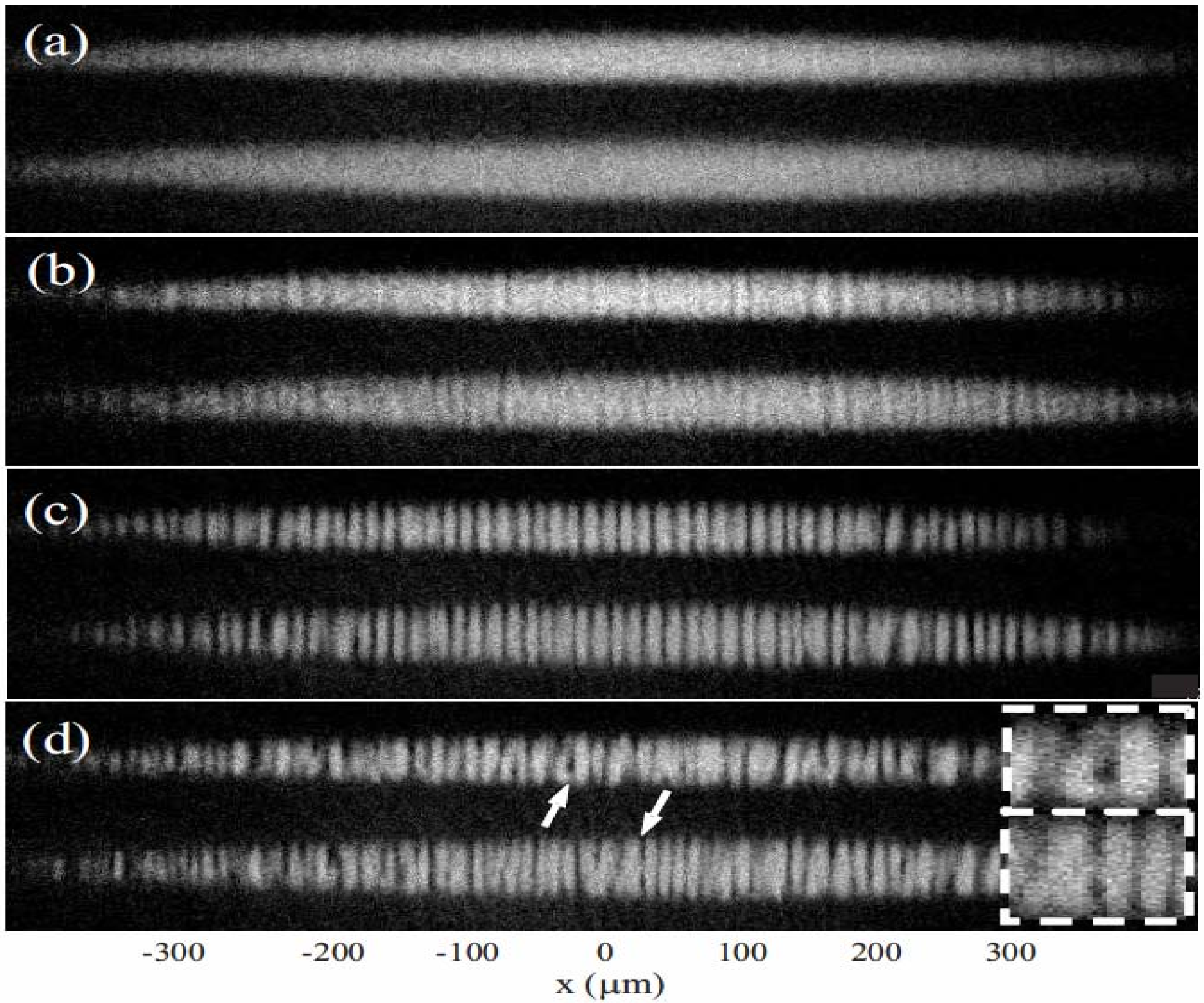
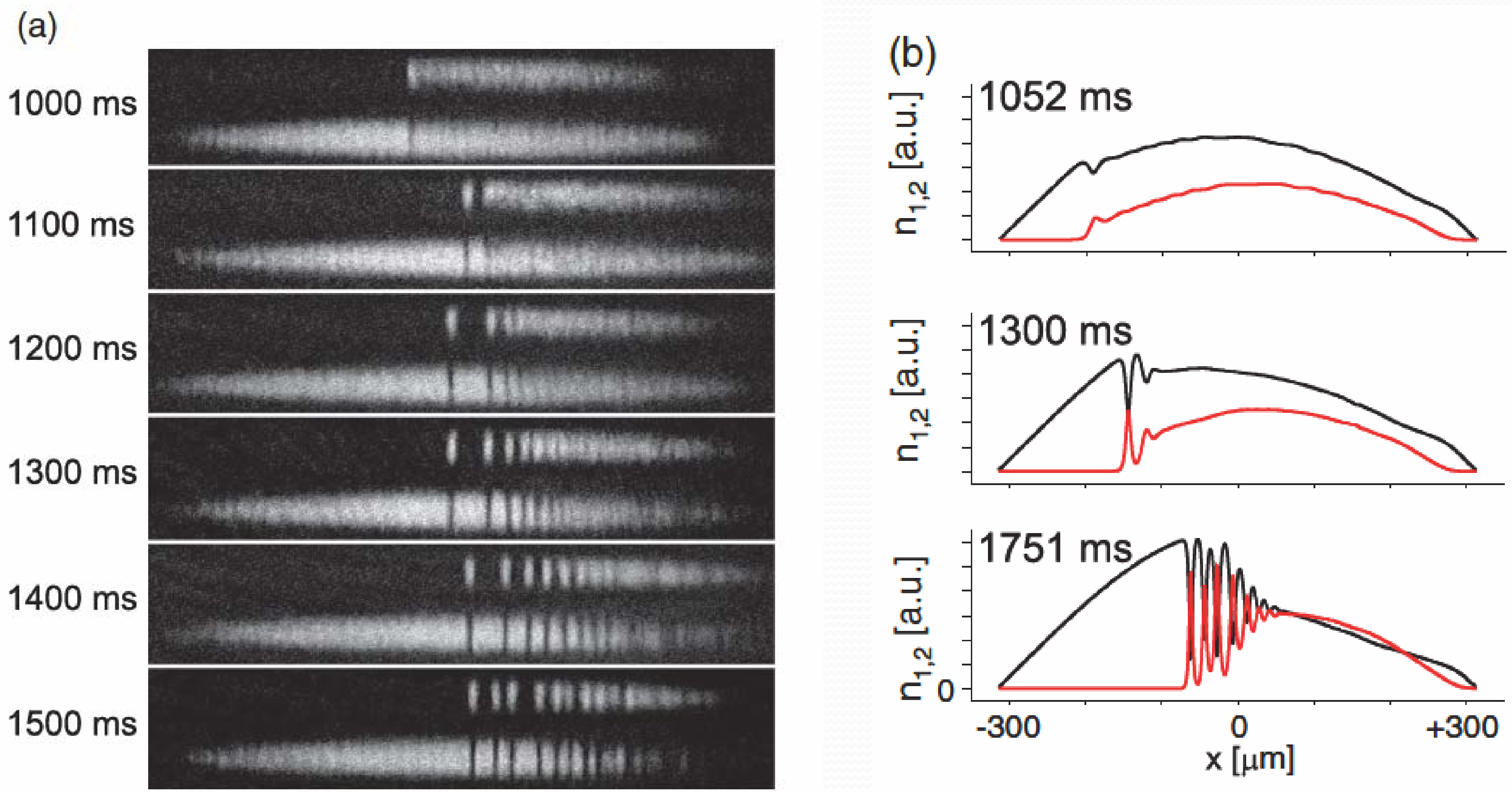
Experiment (P.Engels et al, PRA, 84, 041605 (2011))

Generation of polarization waves by counter flows of two components



Experiment (C.Hamner et al, PRL, 106, 065302 (2011))



Gross-Pitaevskii equations

$$i\hbar\frac{\partial\Psi_+}{\partial t} + \frac{\hbar^2}{2m}\Delta\Psi_+ - \left(g_{11}|\Psi_+|^2 + g_{12}|\Psi_-|^2\right)\Psi_+ = 0,$$

$$i\hbar\frac{\partial\Psi_-}{\partial t} + \frac{\hbar^2}{2m}\Delta\Psi_- - \left(g_{12}|\Psi_+|^2 + g_{22}|\Psi_-|^2\right)\Psi_- = 0,$$

$$g_{ij} = 4\pi\hbar^2 a_{ij}/m \qquad g_{12}^2 < g_{11}g_{22}$$

$^{87}\mathrm{Rb}$ $\quad a_{11} = 100.4a_0,\ a_{12} = 98.98a_0 \quad a_{22} = 98.98a_0$

$$i\partial_t\psi_\pm + \frac{1}{2}\partial_{xx}^2\psi_\pm - \sigma\left(|\psi_\pm|^2 + |\psi_\mp|^2\right)\psi_\pm = 0.$$

Density and polarization variables

$$\begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} = \sqrt{\rho}\, e^{i\Phi/2}\chi = \sqrt{\rho}\, e^{i\Phi/2} \begin{pmatrix} \cos\frac{\theta}{2}\, e^{-i\phi/2} \\ \sin\frac{\theta}{2}\, e^{i\phi/2} \end{pmatrix}$$

Densities

$$\rho(x,t) = |\psi_+|^2 + |\psi_-|^2$$

$$\rho_+(x,t) = \rho\cos^2(\theta/2), \quad \rho_-(x,t) = \rho\sin^2(\theta/2)$$

Flow velocities

$$U = \Phi_x \text{ и } v = \phi_x$$

$$\varphi_\pm(x,t) = \tfrac{1}{2}(\Phi \mp \phi), \quad v_\pm(x,t) = \partial_x\varphi_\pm = \tfrac{1}{2}(U \mp v)$$

Dynamics

A.M.K., Europhysics Letters, **103,** 60003 (2013)

$$\rho_t + \tfrac{1}{2}[\rho(U - v\cos\theta)]_x = 0,$$

$$\Phi_t - \frac{\operatorname{ctg}\theta}{2\rho}(\rho\theta_x)_x + \frac{\rho_x^2}{4\rho^2} - \frac{\rho_{xx}}{2\rho} + \tfrac{1}{4}(\Phi_x^2 + \theta_x^2 + \phi_x^2) + 2\sigma\rho = 0,$$

$$\theta_t + \frac{1}{2\rho}[(\rho v\sin\theta)_x + \rho U\theta_x] = 0,$$

$$\phi_t - \frac{1}{2\rho\sin\theta}(\rho\theta_x)_x + \tfrac{1}{2}Uv = 0.$$

Nonlinear periodic solution

Separation of variables. Total density wave:

$$\rho = \rho(\xi), \quad \theta = \theta(\xi), \quad \Phi = -2\mu t + \Phi_0(\xi), \quad \phi = \phi(\xi), \quad \text{где} \quad \xi = x - Vt.$$

$$\rho_\xi^2 = 4\mathcal{R}(\rho),$$

$$\mathcal{R}(\rho) = \sigma\rho^3 - (2\mu + V^2)\rho^2 + D\rho - (A^2 + C^2)$$

Polarization wave:

$$\left(\frac{d\theta}{d\xi}\right)^2 = \frac{4}{\rho^2}\left(C^2 - \frac{(B - A\cos\theta)^2}{\sin^2\theta}\right)$$

Periodic solution

Total density

$$\rho(\xi) = \rho_1 + (\rho_2 - \rho_1)\mathrm{sn}^2(\sqrt{\rho_3 - \rho_1}\,(\xi + \xi_0), m)$$

$$m = (\rho_2 - \rho_1)/(\rho_3 - \rho_1)$$

Polarization wave

$$R^2 \equiv \rho_1\rho_2\rho_3 \qquad A = R\cos\gamma, \quad C = R\sin\gamma, \quad B = R\cos\beta.$$

$$\theta_1 = \beta + \gamma, \quad \theta_2 = \beta - \gamma$$

$$\cos\theta(\xi) = \cos\theta_1 \sin^2\frac{X(\xi)}{2} + \cos\theta_2\cos^2\frac{X(\xi)}{2}$$

$$X(\xi) = 2R\int_{\xi_0}^{\xi}\frac{d\xi'}{\rho_1 + (\rho_2 - \rho_1)\mathrm{sn}^2(\sqrt{\rho_3 - \rho_1}\,\xi', m)} + X_0$$

Polarization wave

For the case of equal counter flows we get:

$$\cos\theta = \sin\gamma \cdot \cos\left[\frac{2V}{\cos\gamma}(x - Vt)\right]$$

$$k = 2V/\cos\gamma, \quad \omega(k) = \frac{1}{2}k^2\cos\gamma$$

Comparison with experiment (dimensional units). Wavelength equals to

$$\Lambda = \frac{\pi\xi_{1D}c_s}{V}$$

$$\xi_{1D} = \hbar/\sqrt{mg_{1D}\rho_{ch}} \approx 2.4 \times 10^{-5}\ \text{cm}$$

$$c_s = \sqrt{g_{1D}\rho_{ch}/m} \approx 0.3\ \text{cm s}^{-1}$$

Experimental velocity $V \approx 1.8 \times 10^{-2}\ \text{cm s}^{-1}$ corresponds to $\Lambda \approx 13\ \mu\text{m}$

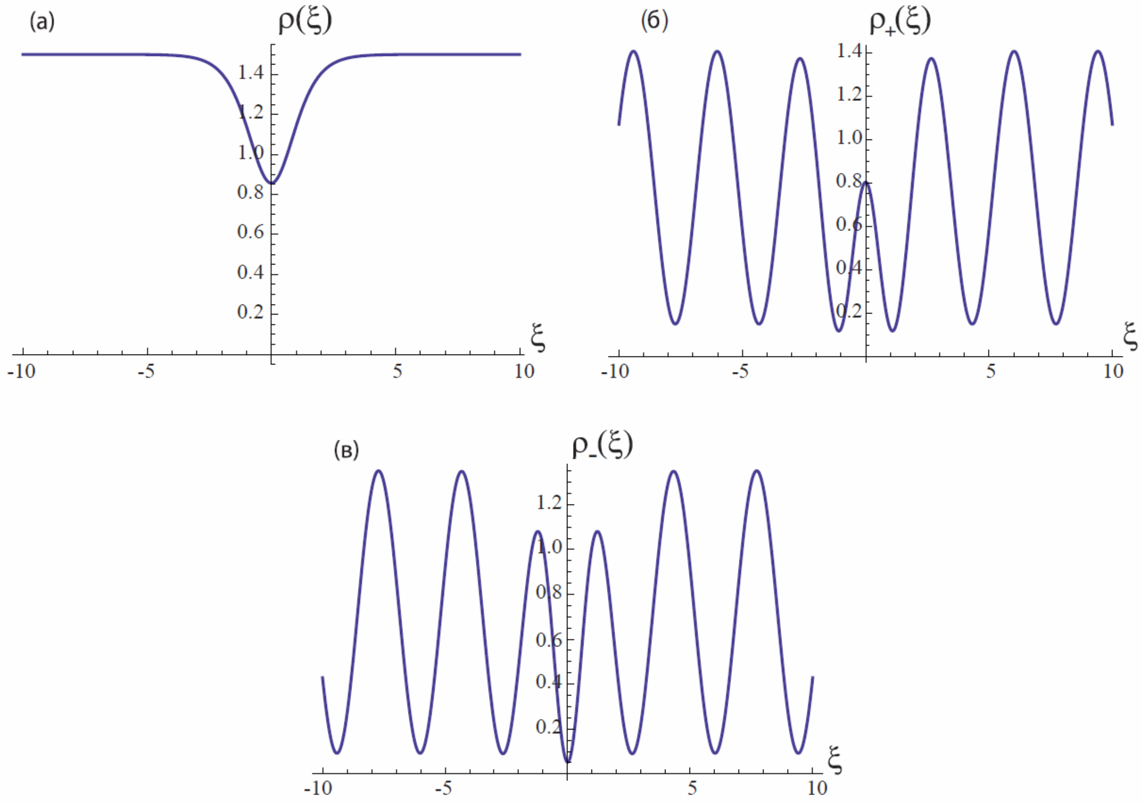
Experimental value of wavelength $\Lambda \approx 15\text{–}18\ \mu\text{m}$

Quasi-soliton

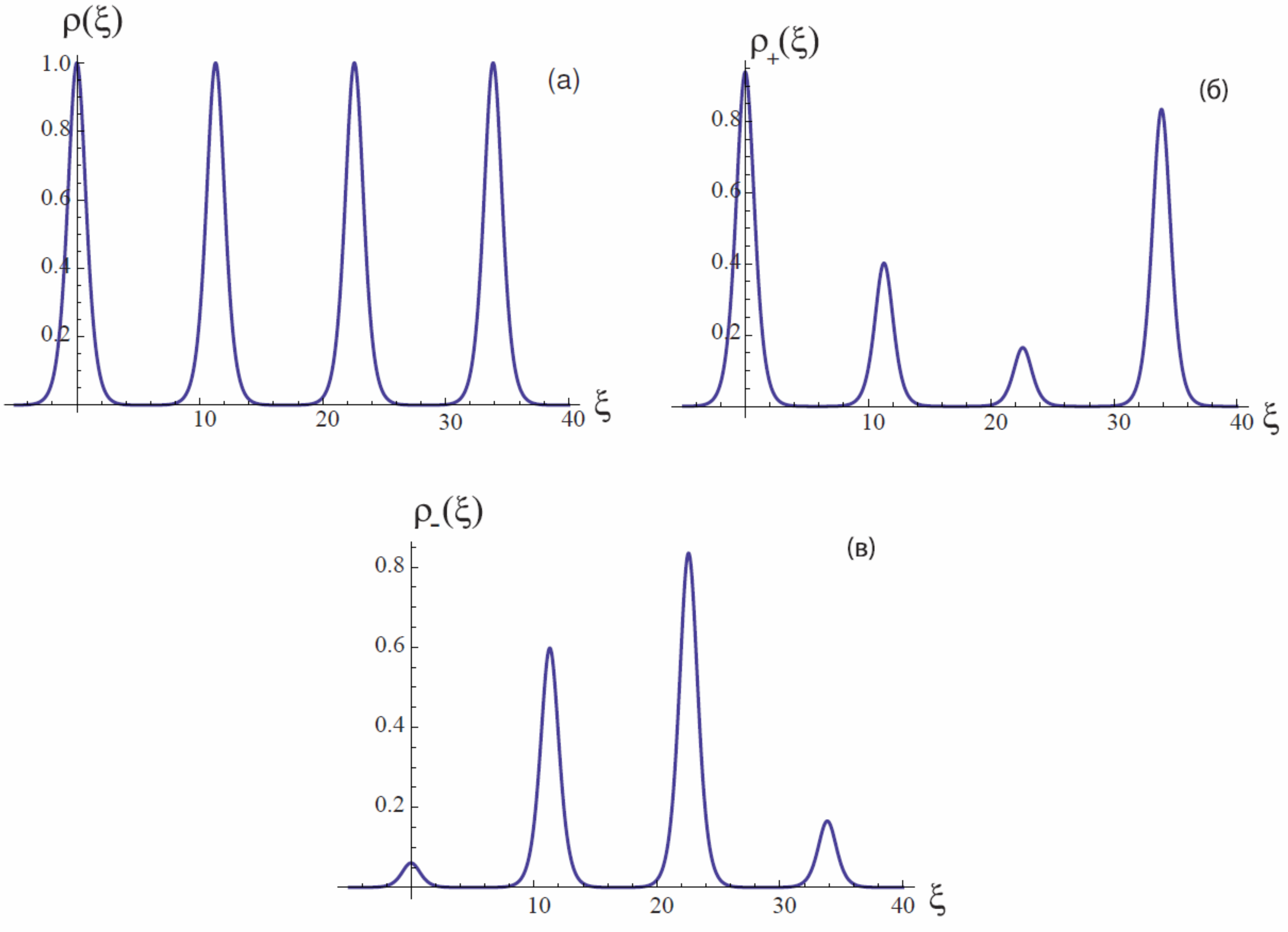
$$\rho_2 = \rho_3 = \rho_0$$

$$\rho(x,t) = \rho_0 \left\{ 1 - \frac{1 - \rho_1/\rho_0}{\mathrm{ch}^2[\sqrt{\rho_0 - \rho_1}(x - Vt)]} \right\}$$

$$X(\xi) = 2\sqrt{\rho_1}\,\xi + 2\,\mathrm{arctg}\left[\sqrt{\rho_0/\rho_1 - 1}\,\mathrm{th}(\sqrt{\rho_0 - \rho_1}\,\xi)\right]$$



Condensate with attractive interaction between atoms



Nonequal nonlinear constants

A.M.K., Y. V. Kartashov, P.-É. Larré, N. Pavloff, Phys. Rev. A **89**, 033618 (2014)

$$i\partial_t\psi_\pm + \tfrac{1}{2}\partial_{xx}^2\psi_\pm - \left[(\alpha_1 \pm \delta)|\psi_\pm|^2 + \alpha_2|\psi_\mp|^2\right]\psi_\pm = 0,$$

$$\rho_t + \tfrac{1}{2}[\rho(U - v\cos\theta)]_x = 0,$$

$$\Phi_t - \frac{\cot\theta}{2\rho}(\rho\theta_x)_x + \frac{\rho_x^2}{4\rho^2} - \frac{\rho_{xx}}{2\rho} + \tfrac{1}{4}(\Phi_x^2 + \theta_x^2 + \phi_x^2)$$

$$+ \rho(\alpha_1 + \alpha_2 + \delta\cos\theta) = 0,$$

$$\rho\theta_t + \tfrac{1}{2}[(\rho v\sin\theta)_x + \rho U\theta_x] = 0,$$

$$\phi_t - \frac{1}{2\rho\sin\theta}(\rho\theta_x)_x + \tfrac{1}{2}Uv - \rho[\delta + (\alpha_1 - \alpha_2)\cos\theta] = 0,$$

Linear waves

Dispersion laws

$$\omega_d^2(k) = \frac{1}{2}\rho_0\left(\alpha_1 + \sqrt{\alpha_2^2 + \delta^2}\right)k^2 + \frac{1}{4}k^4,$$

$$\omega_p^2(k) = \frac{1}{2}\rho_0\left(\alpha_1 - \sqrt{\alpha_2^2 + \delta^2}\right)k^2 + \frac{1}{4}k^4,$$

Two sound velocities

$$c_d^2 = \frac{1}{2}\rho_0\left(\alpha_1 + \sqrt{\alpha_2^2 + \delta^2}\right),$$

$$c_p^2 = \frac{1}{2}\rho_0\left(\alpha_1 - \sqrt{\alpha_2^2 + \delta^2}\right).$$

Weakly nonlinear density waves

In quadratic with respect to amplitude approximation we get the KdV equation

$$\rho'_t + c_d\rho'_x + \frac{3(2\sqrt{\alpha_2^2+\delta^2}-\alpha_2)c_d}{2\rho_0\sqrt{\alpha_2^2+\delta^2}}\rho'\rho'_x - \frac{1}{8c_d}\rho'_{xxx} = 0,$$

For

$$\delta \to 0$$

we obtain

$$\rho'_t + c_d\rho'_x + \frac{3c_d}{2\rho_0}\rho'\rho'_x - \frac{1}{8c_d}\rho'_{xxx} = 0, \quad c_d = \sqrt{\tfrac{1}{2}\rho_0(\alpha_1+\alpha_2)}$$

Weakly nonlinear polarization waves

In quadratic with respect to amplitude approximation we get the KdV equation

$$\theta'_t + c_p\theta'_x + \frac{3c_p(2\delta^2 + \alpha_2^2 - \alpha_2\sqrt{\alpha_2^2 + \delta^2})}{2\delta\sqrt{\alpha_2^2 + \delta^2}}\theta'\theta'_x - \frac{1}{8c_p}\theta'_{xxx} = 0$$

For $\delta \to 0$

$$\theta'_t + c_p\theta'_x + \frac{9c_p\delta}{4\alpha_2}\theta'\theta'_x - \frac{1}{8c_p}\theta'_{xxx} = 0, \quad c_p = \sqrt{\frac{\rho_0(\alpha_1 - \alpha_2)}{2}}$$

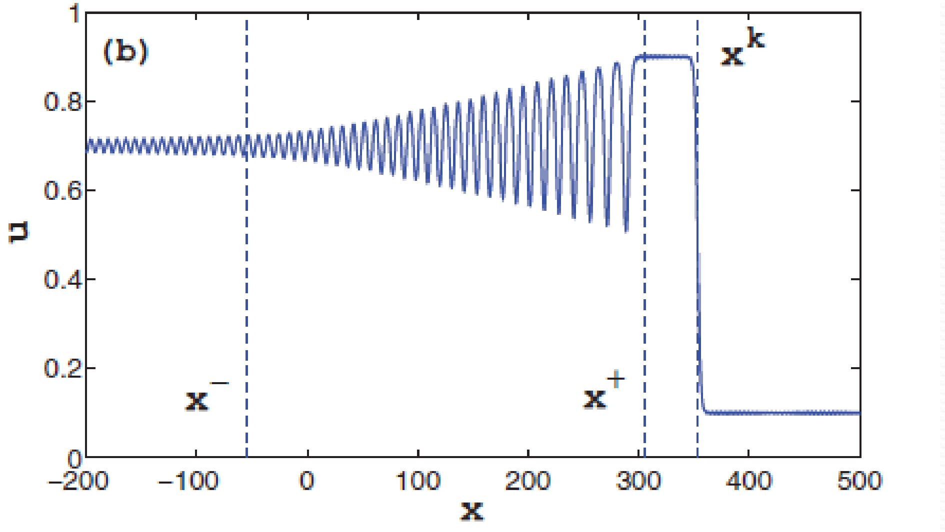
In cubic with respect to amplitude approximation we get the Gardner equation

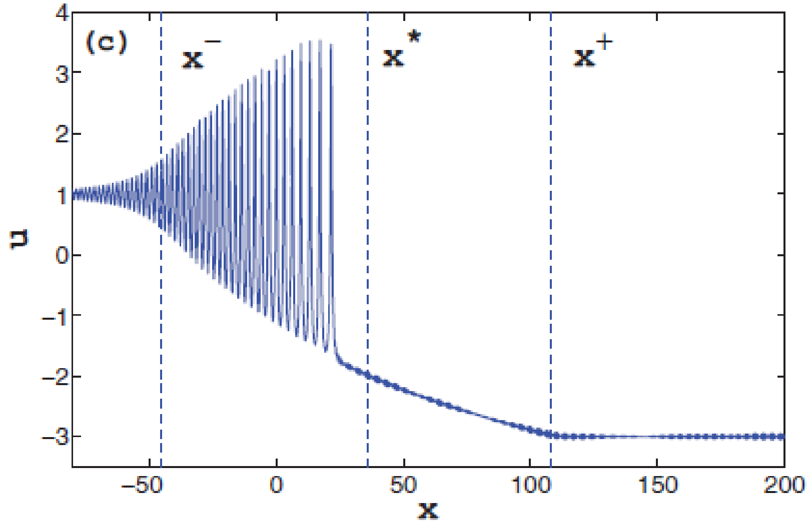
$$\theta'_t + \left(c_p - \frac{\rho_0\delta^2}{8c_p\alpha_2}\right)\theta'_x + \frac{9\,c_p\,\delta}{4\alpha_2}\theta'\theta'_x - \frac{3c_p}{8\alpha_2}(9\alpha_1 - \alpha_2)\,\theta'^2\theta'_x - \frac{1}{8c_p}\theta'_{xxx} = 0$$

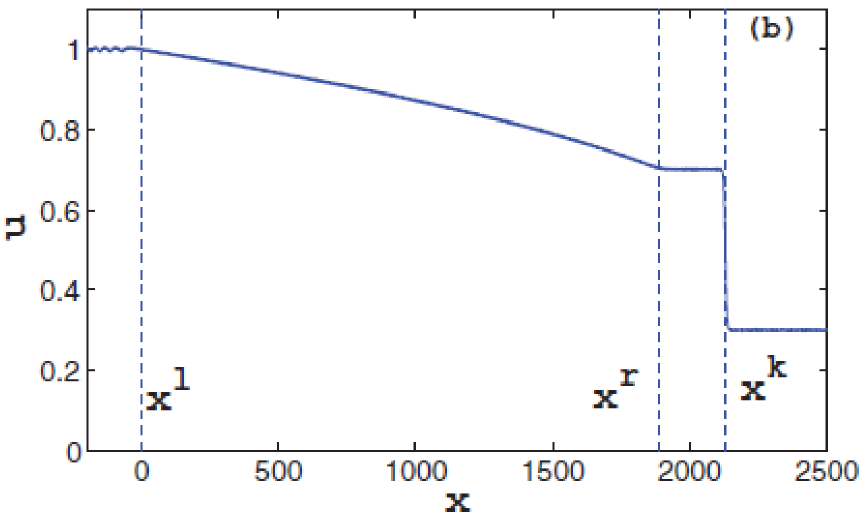
Dispersive shock waves for Gardner equation

A.M.K., Y.-H. Kuo, T.-C. Lin et al., Phys. Rev. E **89**, 036605 (2012)

$$u_t + 6u(1 - \alpha u)u_x = 0,$$







Breather solution

(R. Grimshaw, D. Pelinovsky, E. Pelinovsky, and T. Talipova,Physica D **159**, 35 (2001))

$$\theta'(x,t) = -\frac{2}{c_p}\sqrt{\frac{2\alpha_2}{9\alpha_1-\alpha_2}} \times \frac{\partial}{\partial x}\arctan\frac{\kappa\cosh p\cos\Theta_b - k\cos q\sinh\Phi_b}{\kappa\sinh p\sin\Theta_b + k\sin q\cosh\Phi_b}$$

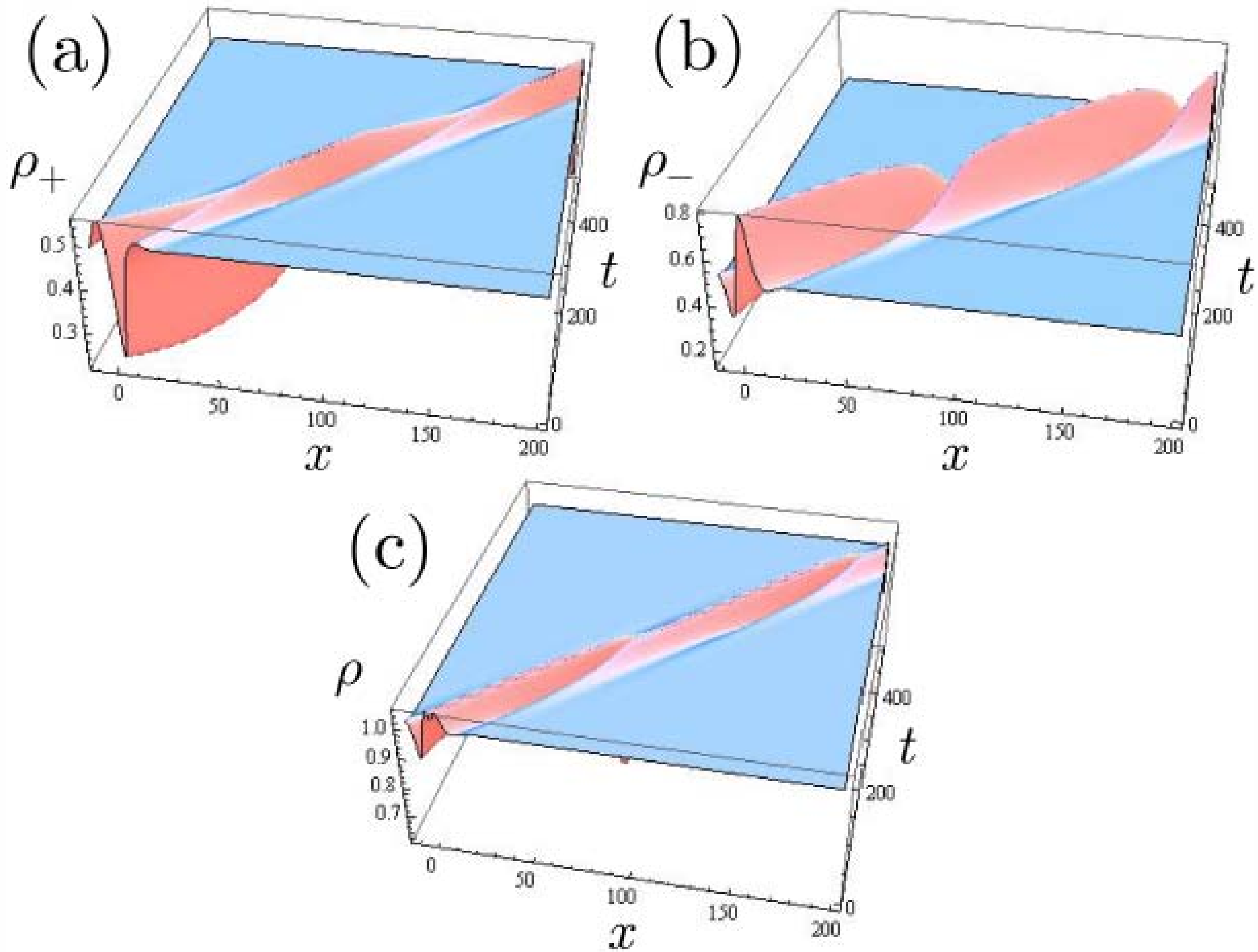
$$\Theta_b = k(v - V_b t) + \Theta_0, \quad \Phi_b = \kappa(x - V_i t) + \Phi_0.$$

$$k = \frac{3\delta c_p}{\sqrt{2\alpha_2(9\alpha_1-\alpha_2)}}\frac{\sinh(2p)}{\cos^2 q\cosh^2 p + \sin^2 q\sinh^2 p},$$

$$\kappa = \frac{3\delta c_p}{\sqrt{2\alpha_2(9\alpha_1-\alpha_2)}}\frac{\sin(2q)}{\cos^2 q\cosh^2 p + \sin^2 q\sinh^2 p},$$

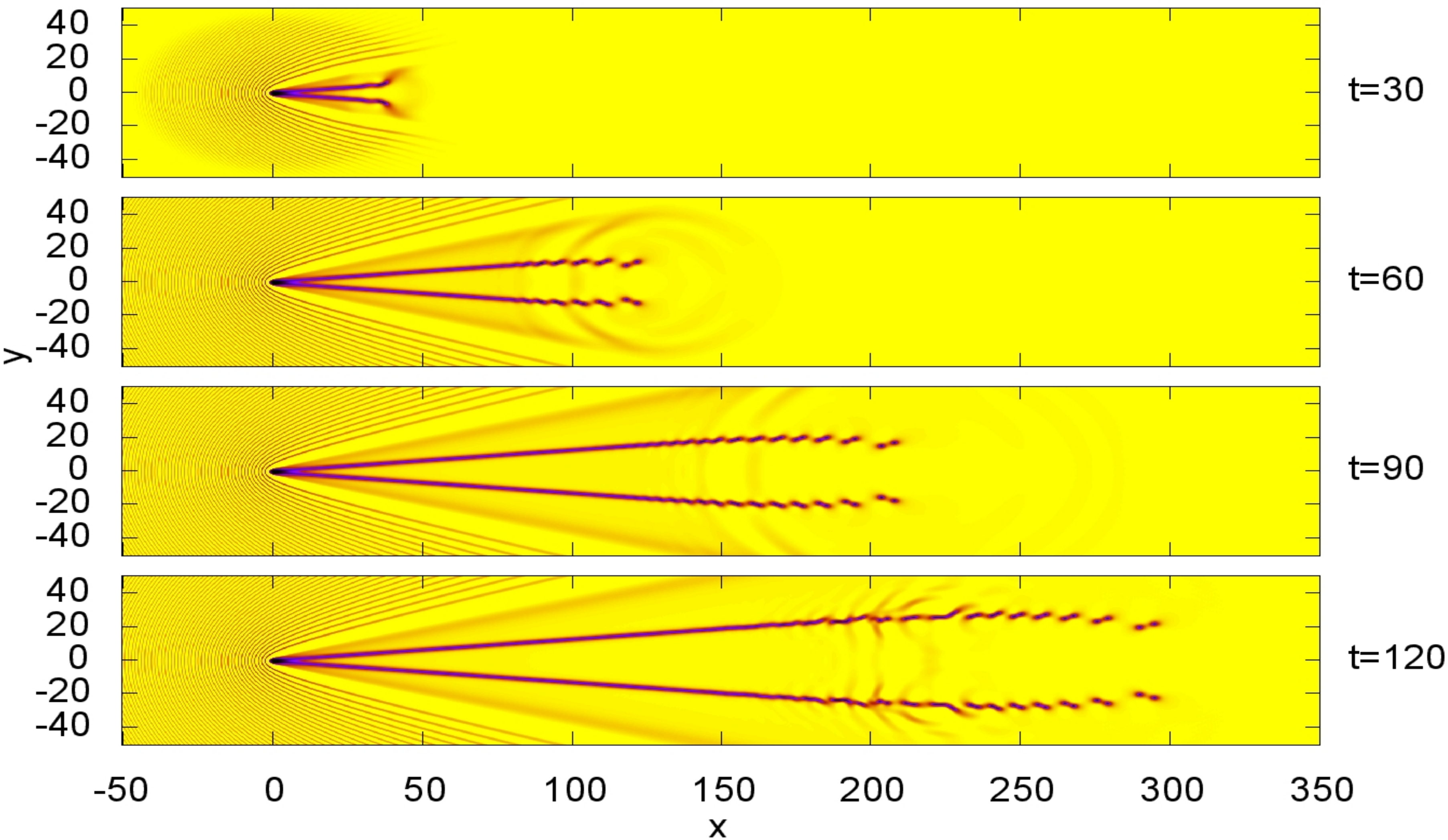
$$V_b = c_p - \frac{\rho_0\delta^2}{8c_p\alpha_2} - \frac{3\kappa^2 - k^2}{8c_p}, \quad V_i = c_p - \frac{\rho_0\delta^2}{8c_p\alpha_2} - \frac{\kappa^2 - 3k^2}{8c_p}$$

Breather

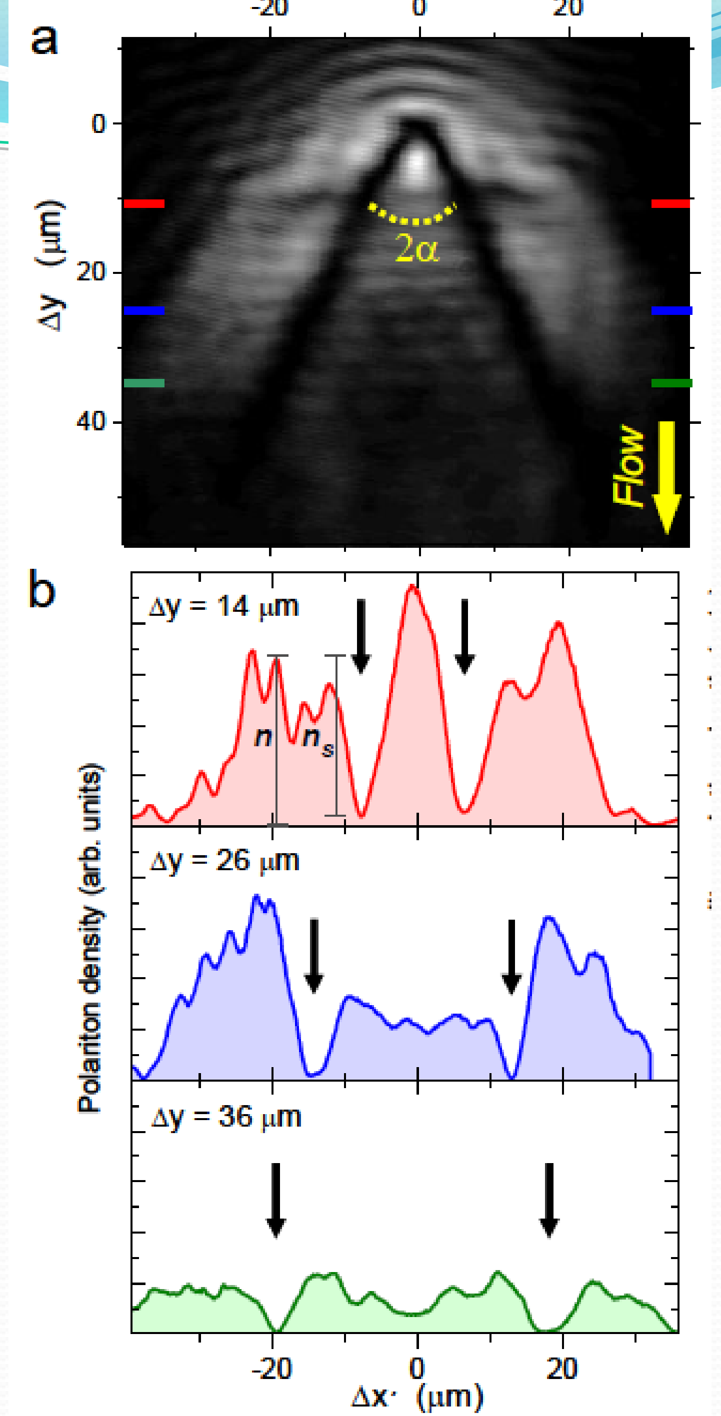

(a)
ρ_+
0.5
0.4
0.3
0
50
100
150
200
x
0
200
400
t
(b)
ρ_-
0.8
0.6
0.4
0.2
0
50
100
150
200
x
0
200
400
t
(c)
ρ
1.0
0.9
0.8
0.7
0
50
100
150
200
x
0
200
400
t

Flow past an obstacle

G.A. El, A. Gammal, A.M.K., PRL, **97,** 180405 (2006)



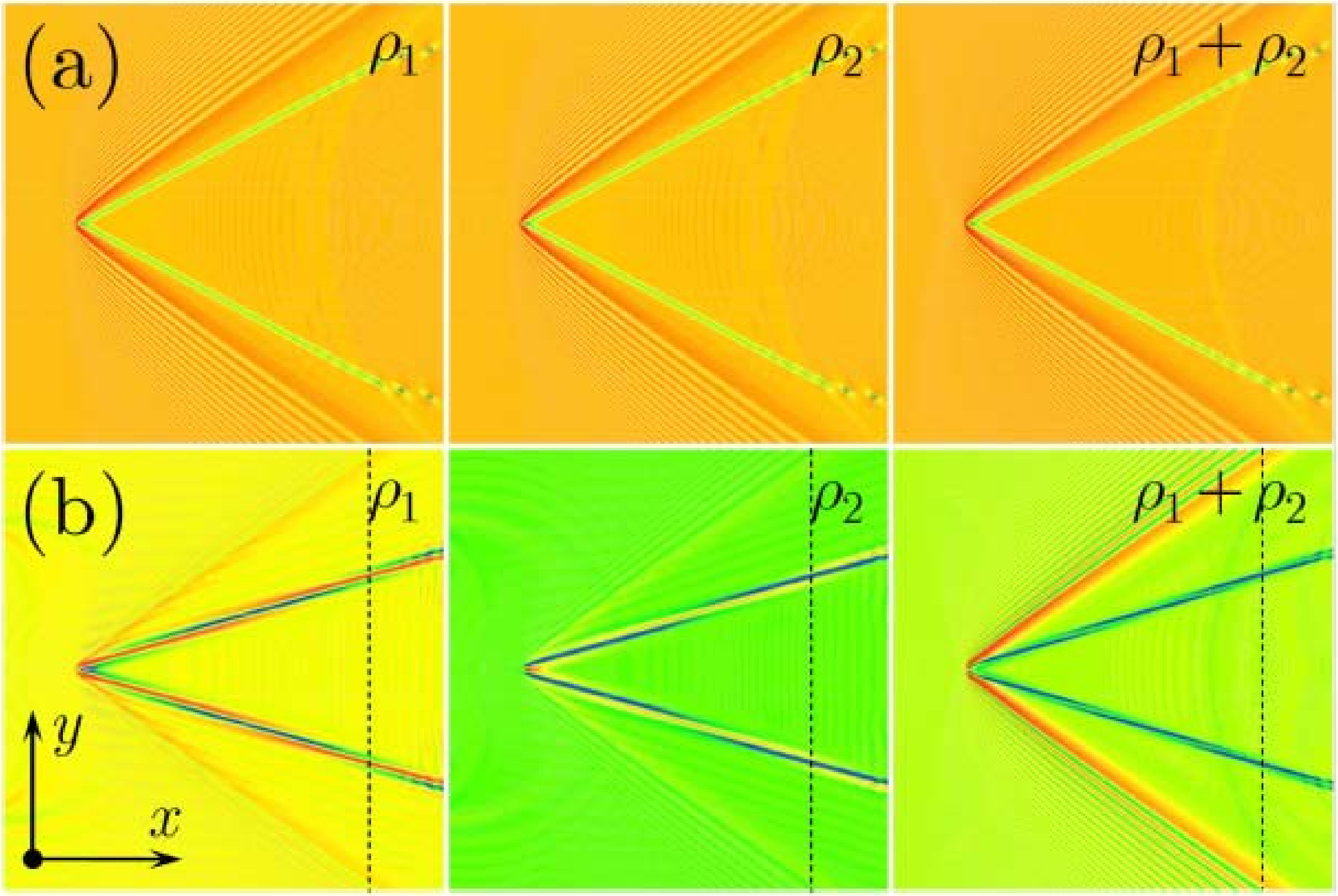
Experiment with a flow past an obstacle of a polariton condensate, A.Amo et al, Science, **332**, 1167 (2011)



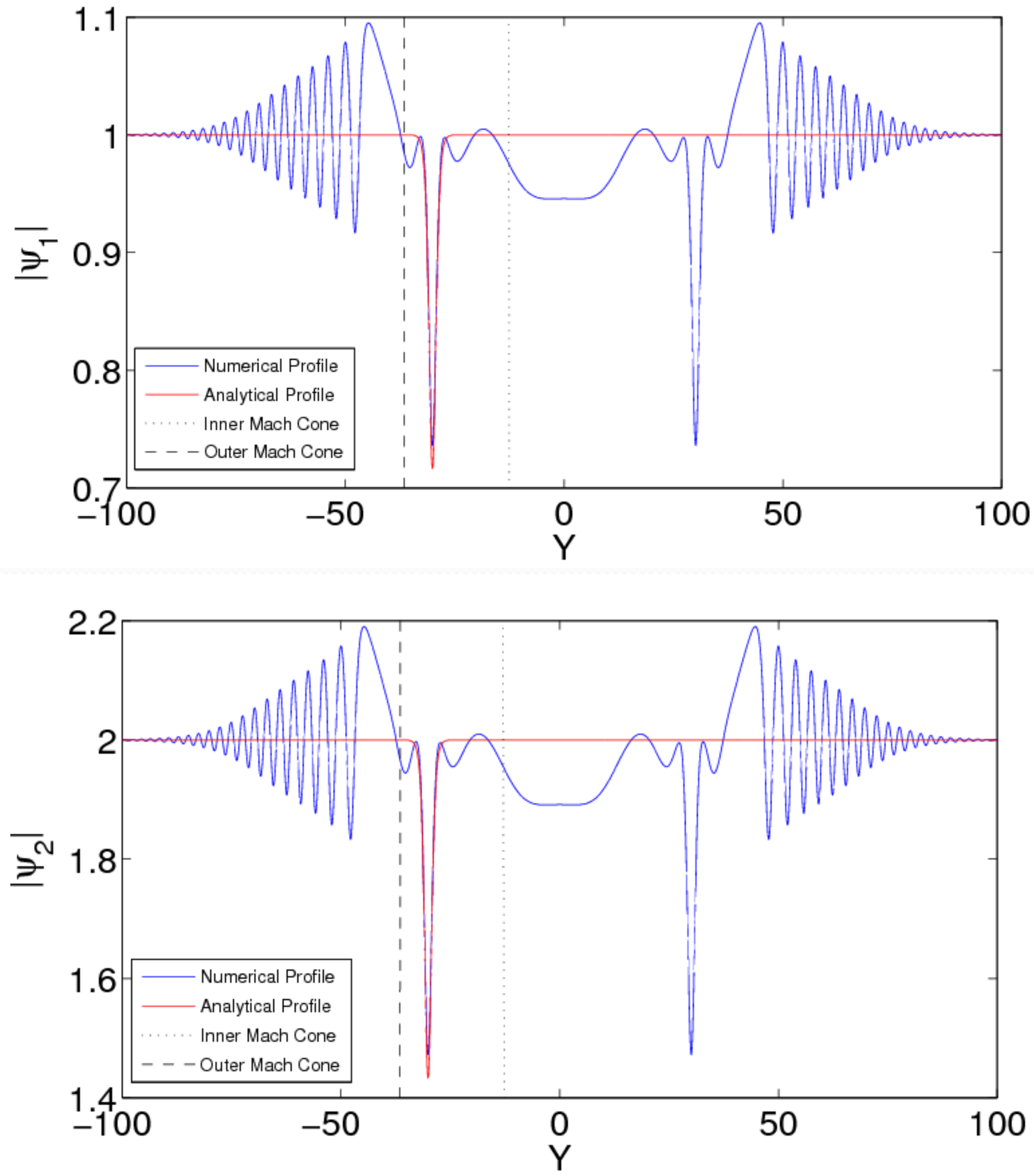
Oblique breather

A.M.K., Y. V. Kartashov, Phys. Rev. Lett. 111, 140402 (2013)

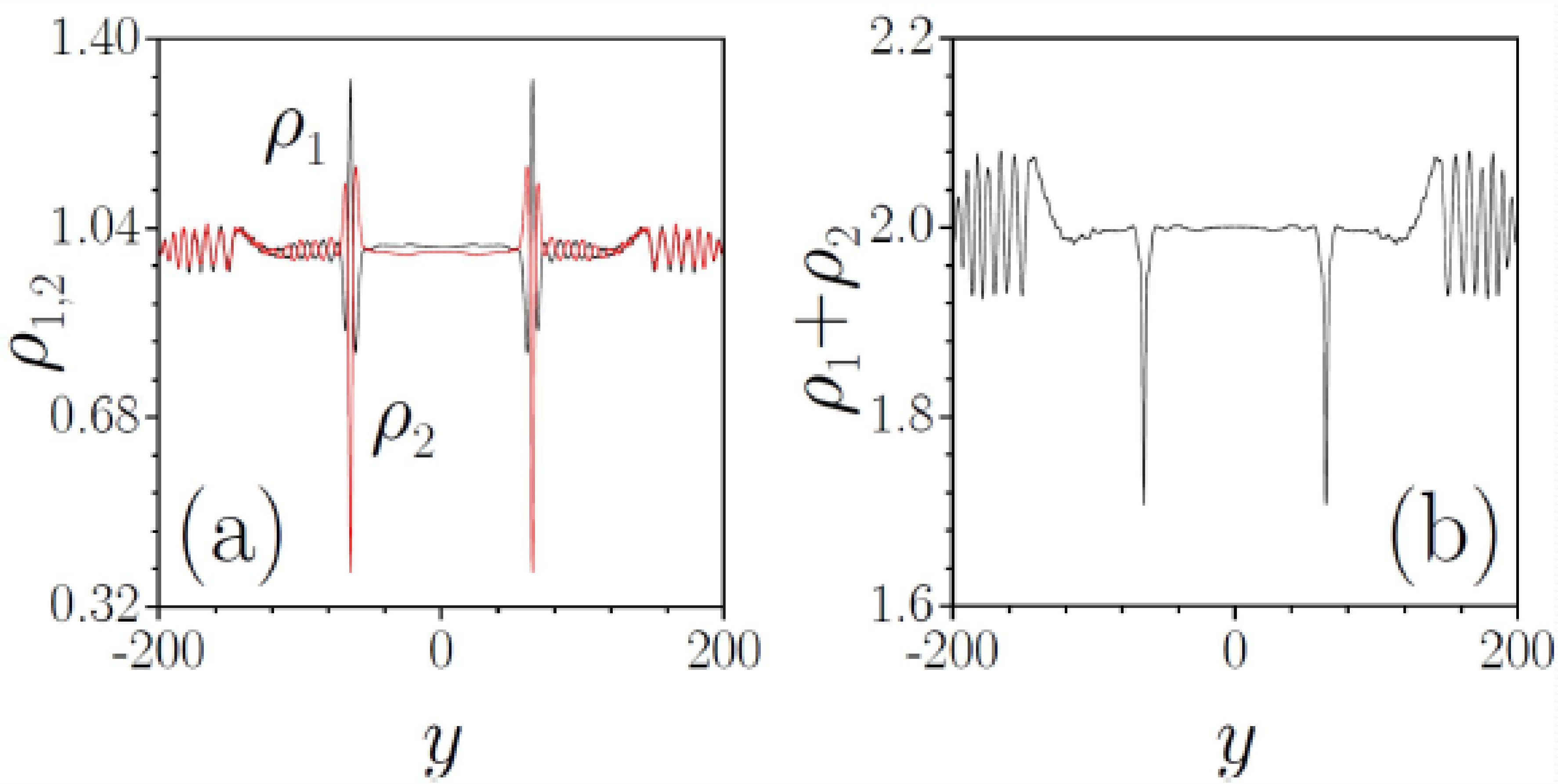
Oblique breathers are generated by the flow of two-component BEC past a polarized obstacle



Non-polarized obstacle



Polarized obstacle



Two-component BEC with spin-orbit coupling

Y. V. Kartashov, A.M.K., EPL, 107, 10008 (2014)

$$i\partial_t\psi_1 + \frac{1}{2}\Delta\psi_1 - (g_{11}|\psi_1|^2 + g_{12}|\psi_2|^2)\psi_1 - i\gamma\partial_y\psi_2 = \kappa_1 V_{\rm obs}(\mathbf{r})\psi_1,$$

$$i\partial_t\psi_2 + \frac{1}{2}\Delta\psi_2 - (g_{12}|\psi_1|^2 + g_{22}|\psi_2|^2)\psi_1 - i\gamma\partial_y\psi_1 = \kappa_2 V_{\rm obs}(\mathbf{r})\psi_2,$$

Dispersion laws of linear waves of density and polarization (equal densities of components)

$$\omega = Vk_x - \gamma k_y \pm c_d^0 k, \quad \omega = Vk_x + \gamma k_y \pm c_p^0 k,$$

Rotation angle

$$\alpha_r = \mp\arctan\left(\frac{\gamma}{V}\right)$$

Non-polarized obstacle

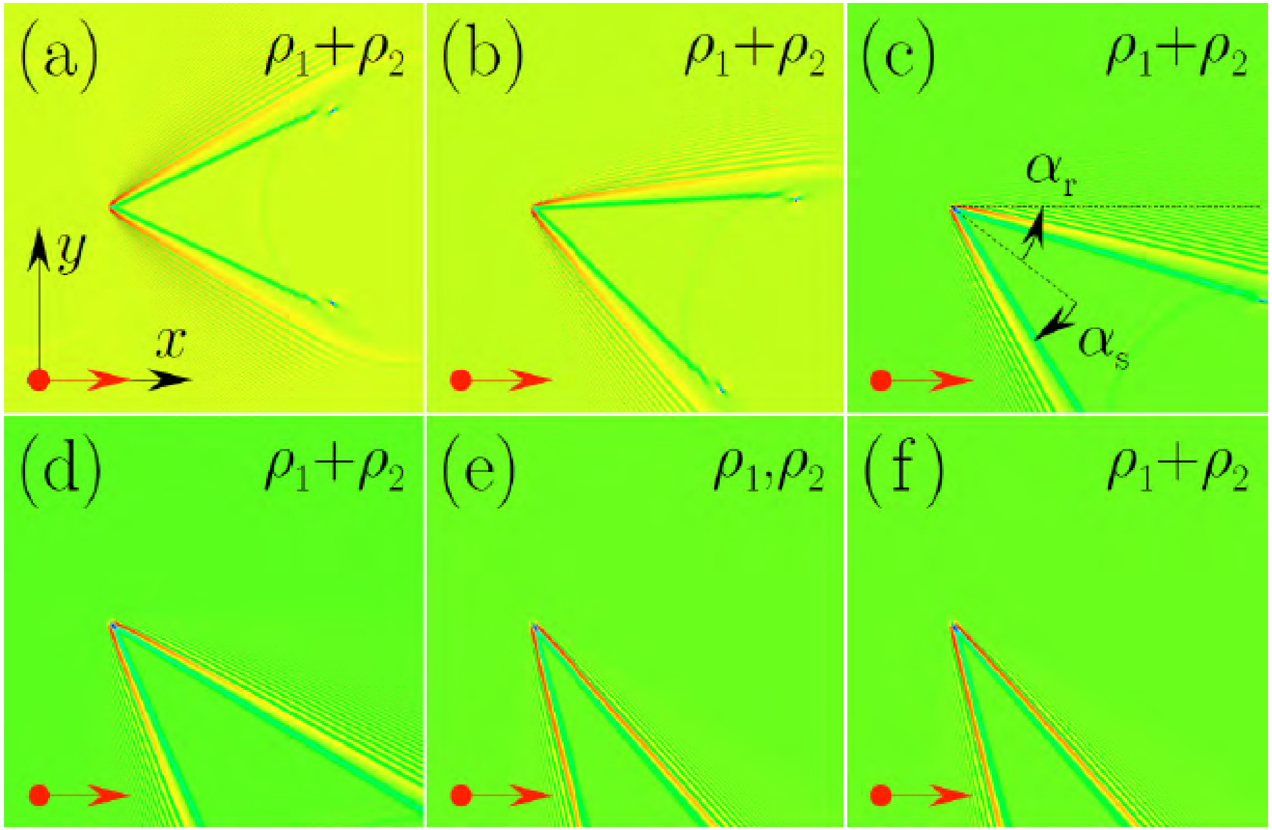
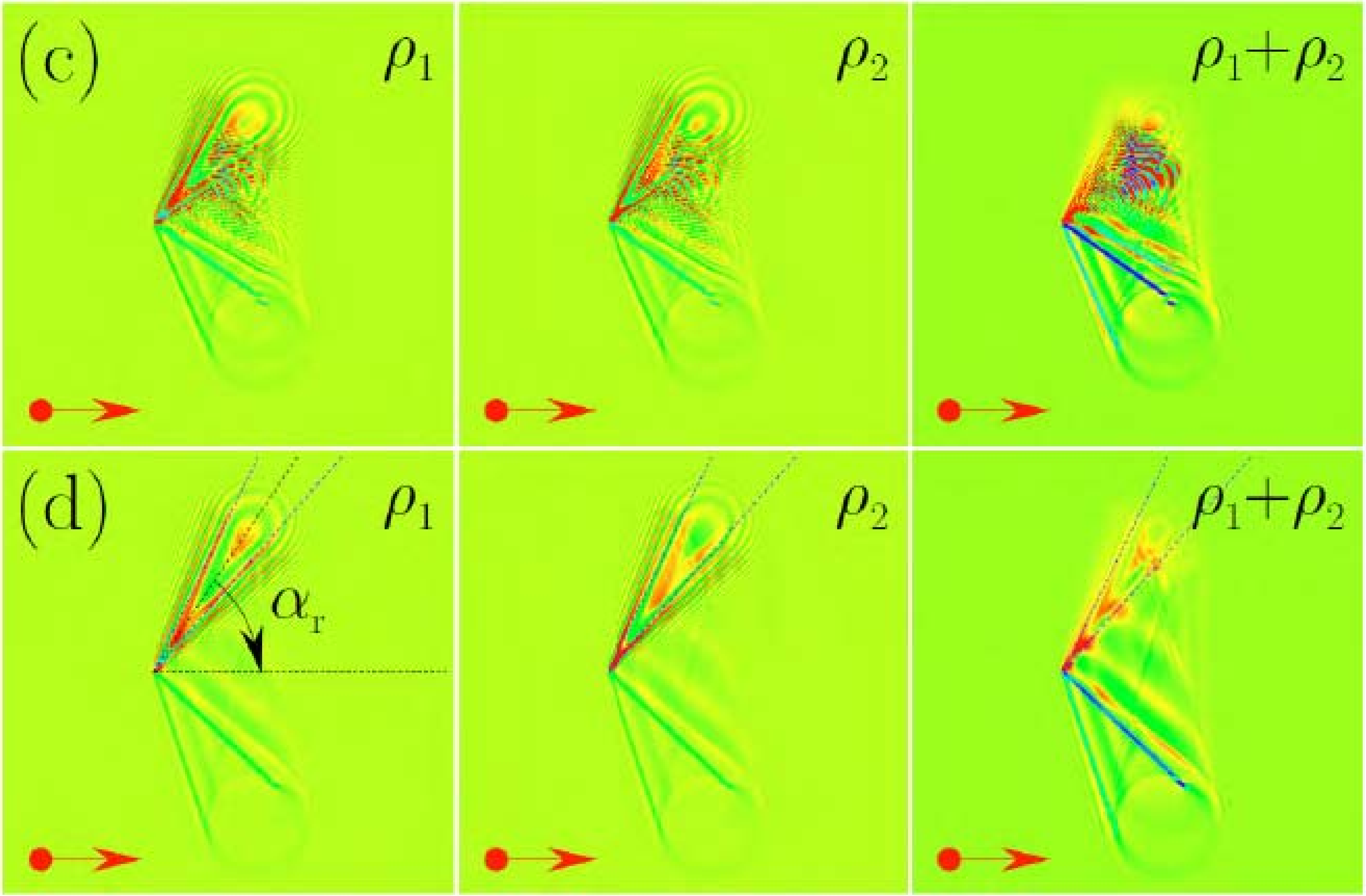


FIG. 1: (Color online) Transverse density distributions at $t = 112$ generated by the flow with velocity $V = 2.6$ past a non-polarized obstacle with $\kappa_1 = \kappa_2 = 1$ at (a) $\gamma = 0$, (b) $\gamma = 1$, (c) $\gamma = 2$, (d) $\gamma = 3$, and (e), (f) $\gamma = 5$. Red arrows indicate the direction of the flow. Rotation angle α_r of the entire pattern and the angle between two oblique dark solitons α_s are indicated on the panel (c).

Polarized obstacle



THANK YOU FOR YOUR ATTENTION!