Darboux transformations with tetrahedral reduction group and nonlocal symmetries

Alexander Mikhailov

University of Leeds

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Plan of the talk

- Algebraic reductions, reduction group and automorphic Lie algebras
- Example: PDEs corresponding to the tetrahedral reduction group
- Elementary Darboux transformations with terahedral reduction symmetry
  - Generic and degenerated Darboux transformations
  - Corresponding differential difference integrable systems
  - Bianchi permutability of Darboux maps and difference systems
- Reduction to a scalar equation - a discrete analogue of Kupershmidt's KdV6 equation
- Non-local symmetries of difference equations

Some of the results were obtained together with G.Berkeley, S.Igonin and P.Xenitidis.

Rational in spectral parameter λ linear problems

(Zakharov, Shabat 1978, Zakharov Mikhailov 1978)
Rational in λ linear problems $\Rightarrow$ integrable systems of PDEs.

$$L(\lambda)\Psi(\lambda)=0, \quad A(\lambda)\Psi(\lambda)=0, \qquad \det\Psi(\lambda)\neq 0,$$

$$L(\lambda)=D_x-U_0-\sum_{k=1}^{n}\frac{U_k}{\lambda-\alpha_k},\ A(\lambda)=D_t-V_0-\sum_{p=1}^{m}\frac{V_p}{\lambda-\beta_p}, \quad U_q,V_q\in Mat_{N^2}(\mathbb{C};x,t).$$

The condition $[L(\lambda),A(\lambda)]=0 \ \Leftrightarrow$ the system of $N^2(n+m+1)$ equations (assuming $\alpha_i\neq\beta_j,\ \alpha_i,\beta_j\in\mathbb{C}$):

$$D_t(U_0)-D_x(V_0)+[U_0,V_0]=0,$$

$$D_t(U_k)+[U_k,V_0+\sum_{p=1}^{m}\frac{V_p}{\alpha_k-\beta_p}]=0, \quad k=1,\dots,n,$$

$$D_x(V_p)-[U_0+\sum_{k=1}^{n}\frac{U_k}{\beta_p-\alpha_k},V_p]=0, \quad p=1,\dots,m.$$

on $N^2(n+m+2)$ functions (entries of U_q,V_q). By a **gauge** transformation one can set $U_0=V_0=0$ and get a well determined system of $N^2(n+m)$ equations. Eigenvalues of U_k and V_p are arbitrary functions of x and t respectively and thus we arrive to a well determined system of $N(N-1)(n+m)$ equations.

✠ *In the case* $N=3, n=m=4$ *it is* 48 *equations.*

More general:

$$L(\lambda) = D_x - U(\lambda), \quad A(\lambda) = D_t - V(\lambda), \qquad U(\lambda), V(\lambda) \in \mathcal{A}(\Gamma) = \mathcal{A} \times \mathcal{R}(\Gamma)$$

where $\mathcal{A}$ is a **simple** Lie algebra and $\mathcal{R}(\Gamma)$ is a ring of meromorphic functions with poles at the set Γ and no other singularities.

A **reduction group** G is a subgroup of $\operatorname{Aut}\mathcal{A}(\Gamma)$, so that $G \subset \operatorname{Aut}\mathcal{A}(\Gamma)$.

Automorphic Lie algebra is $\mathcal{A}(\Gamma)^G \subset \mathcal{A}(\Gamma)$.

In the case of rational in λ Lax operators a group G is finite, the set Γ is a finite union of orbits of the group G and $\operatorname{Aut}\mathcal{A}(\Gamma) \subset \operatorname{Aut}(\mathcal{A} \times \mathbb{C}(\lambda))$

If a finite reduction group G is cyclic and $\Gamma = \{0, \infty\}$, then $\mathcal{A}(\Gamma)^G$ is a graded (Kac-Moody) algebra.

In general, $\mathcal{A}(\Gamma)^G$ is a quasi-graded (or almost-graded in terminology proposed by Krichever and Novikov) Lie algebra.

There is a good progress in classification of automorphic Lie algebras (Bury, AVM, Lombardo, Sanders).

Tetrahedral reduction group

We consider $G \sim \mathbb{T}$ generated by two elements of $\mathrm{Aut}\,(\mathfrak{sl}_3(\mathbb{C}) \times \mathbb{C}(\lambda))$

$$
\begin{aligned}
g_s &: \mathbf{a}(\lambda) \to \mathbf{Q}_s \mathbf{a}(\sigma_s^{-1}(\lambda)) \mathbf{Q}_s^{-1} \\
g_r &: \mathbf{a}(\lambda) \to \mathbf{Q}_r \mathbf{a}(\sigma_r^{-1}(\lambda)) \mathbf{Q}_r^{-1} \\
\sigma_s(\lambda) &= \omega\lambda\ , \sigma_r(\lambda) = \frac{\lambda+2}{\lambda-1}
\end{aligned}
$$

$$
\mathbf{Q}_s = \begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \mathbf{Q}_r = \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}
$$

where $\omega = e^{\frac{2\pi i}{3}}$. We have $g_s^3 = g_r^2 = (g_s g_r)^3 = \mathrm{id}$.

There are two smallest orbits $\Gamma_1 = \{1, \omega, \omega^2, \infty\}$ and $\Gamma_0 = \{-2, -2\omega, -2\omega^2, 0\}$.

Automorphic Lie algebra $\mathcal{A}(\Gamma_1)^G$ has a quasi-graded structure

$$
\mathcal{A}(\Gamma_1)^G = \bigoplus_{k=0}^{\infty} \mathcal{A}^k, \quad \mathcal{A}^k = \{J^k \mathbf{a}_1, J^k \mathbf{a}_2, \ldots, J^k \mathbf{a}_8\}, \quad [\mathcal{A}^n, \mathcal{A}^m] \subset \mathcal{A}^{n+m} \bigoplus \mathcal{A}^{n+m+1}
$$

$$
\begin{aligned}
&\mathbf{a}_1 =< \lambda \mathbf{e}_{13} >_{\mathbb{T}},\ \mathbf{a}_2 =< \lambda \mathbf{e}_{21} >_{\mathbb{T}},\ \mathbf{a}_3 =< \lambda \mathbf{e}_{32} >_{\mathbb{T}} \\
&\mathbf{a}_4 = [\mathbf{a}_1, \mathbf{a}_3],\ \mathbf{a}_5 = [\mathbf{a}_2, \mathbf{a}_1],\ \mathbf{a}_6 = [\mathbf{a}_3, \mathbf{a}_2] \\
&\mathbf{a}_7 = [[\mathbf{a}_1, \mathbf{a}_3], \mathbf{a}_2],\ \mathbf{a}_8 = [[\mathbf{a}_2, \mathbf{a}_1], \mathbf{a}_3], \qquad J = \frac{\lambda^3(\lambda^3+8)^3}{4(\lambda^3-1)^3}
\end{aligned}
$$

Tetrahedral reduction group

$$\mathbf{a}_1 =< \lambda \mathbf{e}_{13} >_{\mathbb{T}} = \begin{pmatrix} -\frac{1}{6}\frac{\lambda^3+2}{\lambda^3-1} & -\frac{1}{2}\frac{\lambda^2}{\lambda^3-1} & \frac{1}{4}\frac{\lambda^4}{\lambda^3-1} \\ \frac{\lambda}{\lambda^3-1} & \frac{1}{3}\frac{\lambda^3+2}{\lambda^3-1} & -\frac{1}{2}\frac{\lambda^2}{\lambda^3-1} \\ \frac{\lambda^3+2}{\lambda^3-1} & \frac{\lambda}{\lambda^3-1} & -\frac{1}{6}\frac{\lambda^3+2}{\lambda^3-1} \end{pmatrix}$$

$$\mathbf{a}_2 =< \lambda \mathbf{e}_{21} >_{\mathbb{T}} = \begin{pmatrix} -\frac{1}{6}\frac{\lambda^3+2}{\lambda^3-1} & \frac{\lambda^2}{\lambda^3-1} & \frac{\lambda}{\lambda^3-1} \\ \frac{1}{4}\frac{\lambda^4}{\lambda^3-1} & -\frac{1}{6}\frac{\lambda^3+2}{\lambda^3-1} & -\frac{1}{2}\frac{\lambda^2}{\lambda^3-1} \\ -\frac{1}{2}\frac{\lambda^2}{\lambda^3-1} & \frac{\lambda}{\lambda^3-1} & \frac{1}{3}\frac{\lambda^3+2}{\lambda^3-1} \end{pmatrix}$$

$$\mathbf{a}_3 =< \lambda \mathbf{e}_{32} >_{\mathbb{T}} = \begin{pmatrix} \frac{1}{3}\frac{\lambda^3+2}{\lambda^3-1} & -\frac{1}{2}\frac{\lambda^2}{\lambda^3-1} & \frac{\lambda}{\lambda^3-1} \\ \frac{\lambda}{\lambda^3-1} & -\frac{1}{6}\frac{\lambda^3+2}{\lambda^3-1} & \frac{\lambda^2}{\lambda^3-1} \\ -\frac{1}{2}\frac{\lambda^2}{\lambda^3-1} & \frac{1}{4}\frac{\lambda^4}{\lambda^3-1} & -\frac{1}{6}\frac{\lambda^3+2}{\lambda^3-1} \end{pmatrix}.$$

$$\mathbf{b}_1 =< \lambda^{-1} \mathbf{e}_{12} >_{\mathbb{T}}, \ \mathbf{b}_2 =< \lambda^{-1} \mathbf{e}_{23} >_{\mathbb{T}}, \ \mathbf{b}_3 =< \lambda^{-1} \mathbf{e}_{31} >_{\mathbb{T}}$$

$$\mathbf{b}_1 = \begin{pmatrix} \frac{\frac{-1}{6}\lambda^3+\frac{2}{3}}{\lambda^3+8} & \frac{2}{\lambda^4+8\lambda} & \frac{-\lambda}{\lambda^3+8} \\ 2\frac{\lambda}{\lambda^3+8} & \frac{\frac{-1}{6}\lambda^3+\frac{2}{3}}{\lambda^3+8} & \frac{-\lambda^2}{\lambda^3+8} \\ \frac{-\lambda^2}{\lambda^3+8} & \frac{-\lambda}{\lambda^3+8} & \frac{\frac{1}{3}\lambda^3-\frac{4}{3}}{\lambda^3+8} \end{pmatrix}, \ \ldots$$

Tetrahedral reduction group

$$L = \partial_x + u\mathbf{a}_1 + v\mathbf{a}_2 + w\mathbf{a}_3, \qquad uvw = 1,$$

$$A = \partial_t + \sum_{i=1}^{3} p_i \mathbf{a}_i + \sum_{i=4}^{6} q_i \mathbf{a}_i.$$

$$[L, A] = 0 \;\Leftrightarrow\; \begin{cases} i\psi_t = \psi_{xx} + \bar{\psi}_x^2 + i\left(e^{\psi+\bar{\psi}} + \omega^* e^{\omega\psi+\omega^*\bar{\psi}} + \omega e^{\omega^*\psi+\omega\bar{\psi}}\right)\bar{\psi}_x \\ -i\bar{\psi}_t = \bar{\psi}_{xx} + \psi_x^2 - i\left(e^{\psi+\bar{\psi}} + \omega e^{\omega\psi+\omega^*\bar{\psi}} + \omega^* e^{\omega^*\psi+\omega\bar{\psi}}\right)\psi_x \end{cases}$$

$$B = \partial_y + p\mathbf{b}_1 + q\mathbf{b}_2 + r\mathbf{b}_3, \qquad pqr = 1$$

$$[L, B] = 0 \;\Leftrightarrow\; \begin{cases} \phi_x = e^{\bar{\phi}} - e^{\bar{\psi}}, \quad \psi_x = e^{-\bar{\phi}-\bar{\psi}} - e^{\bar{\phi}}, \\ \bar{\phi}_y = e^{\phi} - e^{\psi}, \quad \bar{\psi}_y = e^{-\phi-\psi} - e^{\phi}. \end{cases}$$

If we let $x = y$, $\bar{\phi} = \phi$, $\bar{\psi} = \psi$, then

$$\phi_{xx} = e^{2\phi} - e^{-\phi}.$$

Darboux transformations M for Lax operators

$$L = \partial_x + u\mathbf{a}_1 + v\mathbf{a}_2 + w\mathbf{a}_3, \qquad uvw = 1$$

is a mapping of a fundamental solution $L\Psi = 0$ to a fundamental solution $\Psi_1 = M(\lambda)\Psi,\ L_1\Psi_1 = 0$ where

$$L_1 = \partial_x + u_1\mathbf{a}_1 + v_1\mathbf{a}_2 + w_1\mathbf{a}_3, \qquad u_1 v_1 w_1 = 1.$$

Let $L = \partial_x + U,\ L_1 = \partial_x + S(U),\ \Psi_1 = S(\Psi)$, then it follows from $[\partial_x, S] = 0$ that

$$M_x(\lambda) + S(U)M(\lambda) - M(\lambda)U = 0$$

Darboux matrix $M(\lambda)$ inherits symmetries of the Lax operator. There exists such $M(\lambda)$ that

$$M(\lambda) = \mathbf{Q}_s^{-1} M(\omega\lambda)\mathbf{Q}_s, \qquad M(\lambda) = \mathbf{Q}_r^{-1} M\left(\frac{\lambda+2}{\lambda-1}\right)\mathbf{Q}_r.$$

Invariant M with first order poles in λ at Γ is of the form

$$M = \mathbf{I}f + \alpha(uvu_1\mathbf{a}_1 + vu_1v_1\mathbf{a}_2 + \mathbf{a}_3).$$

Where f and α can be found from the condition that $\det M(\lambda)$ is a generating function of first integrals

$$\det M = EJ(\lambda) + F_1 = E(J(\lambda) - \gamma) + F_2$$

$$J(\lambda) = \frac{\lambda^3(\lambda^3+8)^3}{4(\lambda^3-1)^3}, \qquad E = \frac{1}{16}\alpha^3 uv^2u_1^2v_1$$

$$F_1 = \frac{1}{27}(3f+\alpha(1+uvu_1-2vu_1v_1))(3f+\alpha(1-2uvu_1+vu_1v_1))(3f+\alpha(-2+uvu_1+vu_1v_1))$$

$$F_2 = \gamma E + F_1 = \frac{1}{27}(3f+\alpha(1+uvu_1+vu_1v_1))q, \qquad \gamma = J(1+\sqrt{3})$$

where q is an irreducible quadratic polynomial in f. By setting $E = \frac{1}{16}$, $F_1 = c_1 \in \mathbb{C}$ we obtain a generic Darboux matrix parametrised by a constant c_1.

There are four degenerate Darboux matrices (three cases when $F_1 = 0$ and one case when $F_2 = 0$)

$$M_1(u, v, u_1, v_1) = \alpha(-\frac{1}{3}(-2 + uvu_1 + vu_1 v_1)\mathbf{I} + uvu_1\mathbf{a}_1 + vu_1 v_1\mathbf{a}_2 + \mathbf{a}_3)$$

$$M_2(u, v, u_2, v_2) = \beta(-\frac{1}{3}(1 + uvu_2 - 2vu_2 v_2)\mathbf{I} + uvu_2\mathbf{a}_1 + vu_2 v_2\mathbf{a}_2 + \mathbf{a}_3)$$

$$M_3(u, v, u_3, v_3) = \gamma(\frac{1}{3}(-1 + 2uvu_3 - vu_3 v_3)\mathbf{I} + uvu_3\mathbf{a}_1 + vu_3 v_3\mathbf{a}_2 + \mathbf{a}_3)$$

$$M_4(u, v, u_4, v_4) = \delta(-\frac{1}{3}(1 + uvu_4 + vu_4 v_4)\mathbf{I} + uvu_4\mathbf{a}_1 + vu_4 v_4\mathbf{a}_2 + \mathbf{a}_3).$$

$$\alpha^3 uv^2 u_1^2 v_1 = 1, \quad \beta^3 uv^2 u_2^2 v_2 = 1, \quad \gamma^3 uv^2 u_3^2 v_3 = 1, \quad \delta^3 uv^2 u_4^2 v_4 = 1.$$

Elementary Darboux transformations

From $(M_i)_x + S_i(U)M_i - M_iU = 0$ we get corresponding differential-difference systems (or Bäcklund transformations)

$$\begin{pmatrix} -1 & S_1 \\ S_1+1 & 1 \end{pmatrix}\begin{pmatrix} \frac{u_x}{u} \\ \frac{v_x}{v} \end{pmatrix} = \begin{pmatrix} \frac{1}{uv} - \frac{2}{u_1 v} + \frac{1}{u_1 v_1} + u - u_1 - v + v_1 \\ -\frac{1}{uv} + \frac{1}{u_1 v} - u + u_1 \end{pmatrix}$$

$$\begin{pmatrix} -1 & S_2 \\ S_2+1 & 1 \end{pmatrix}\begin{pmatrix} \frac{u_x}{u} \\ \frac{v_x}{v} \end{pmatrix} = \begin{pmatrix} \frac{u_2 v_2}{u} + u - u_2 - v_2 \\ \frac{1}{uv} - \frac{1}{u_2 v_2} - \frac{2u_2 v_2}{u} - u + u_2 + v + v_2 \end{pmatrix}$$

$$\begin{pmatrix} -1 & S_3 \\ S_3+1 & 1 \end{pmatrix}\begin{pmatrix} \frac{u_x}{u} \\ \frac{v_x}{v} \end{pmatrix} = \begin{pmatrix} \frac{uv}{v_3} - u - v + v_3 \\ \frac{uv}{v_3} + \frac{1}{uv} - \frac{1}{u_3 v_3} - u_3 \end{pmatrix}$$

$$\begin{pmatrix} -1 & S_4 \\ S_4+1 & 1 \end{pmatrix}\begin{pmatrix} \frac{u_x}{u} \\ \frac{v_x}{v} \end{pmatrix} = \begin{pmatrix} u - u_4 - v + v_4 \\ \frac{1}{uv} - \frac{1}{u_4 v_4} - u + u_4 \end{pmatrix}$$

Notice that

$$\begin{pmatrix} -1 & S_i \\ S_i+1 & 1 \end{pmatrix}^2 = (1 + S_i + S_i^2)\mathbf{I}.$$

From $[S_i, S_j] = 0$ it follows that $Q_{i,j} = S_i(M_j)M_i - S_j(M_i)M_j = 0$:

$$\begin{cases} -uv - uu_1u_{14}v_4v + uu_4u_{14}v_4v + u_{14}v_4 = 0 \\ -uu_1^2vv_4 + uu_4u_1vv_4 + u_1v_4 + uu_1vv_1v_{14} - uu_1vv_4v_{14} - uv_{14} = 0 \end{cases}$$

$$\begin{cases} -u_{24}u_2^2v_2^2v_4 + uu_{24}u_2^2v_2v_4 + uu_2v_2 + uu_4u_2vv_2v_4 - uu_4u_{24}u_2v_2v_4 - uu_4v_4 = 0 \\ u_2v_2 - uv_{24} - uu_2 + uu_4 = 0 \end{cases}$$

$$\begin{cases} -u_3v_3 + u_3u_4u_{34}v_4v_3 - uu_3u_4vv_4 + u_4v_4 = 0 \\ -u_4v_4 + uv_{34} + v_{34}v_4 - v_3v_{34} = 0 \end{cases}$$

$$\begin{cases} uu_1v_2v^2 - u_2u_{12}v_2^2v - uv - uu_1u_{12}v_2v + uu_2u_{12}v_2v + u_{12}v_2 = 0 \\ -uu_1^2vv_2 - u_2u_1vv_2^2 + uu_2u_1vv_2 + u_1v_2 + uu_1vv_2v_{12} - uv_{12} = 0 \end{cases}$$

$$\begin{cases} u_2u_3u^2vv_2 + u_3uv_3 - u_2u_3uvv_2v_3 - u_2u_3u_{23}uv_2v_3 - uu_2v_2 + u_2^2u_{23}v_2^2v_3 = 0 \\ uv_{23} - u_2v_2 = 0 \end{cases}$$

$$\begin{cases} u_{13}v_3 - uv = 0 \\ -u_1u^2vv_{13} + u_1^2uvv_1 - u_1uvv_1v_{13} + u_1uvv_3v_{13} + uv_{13} - u_1v_3 = 0 \end{cases}$$

The above differential-difference equations are non-local symmetries of these difference systems. Indeed,

$$\partial_x Q_{i,j} = -S_i(S_j(U))Q_{i,j} + Q_{i,j}U = 0.$$

A discrete analogue of Kupershmidt's KdV6 equation

$$(D_x^3 + 8u_x D_x + 4u_{xx})(u_t + u_{xxx} + 6u_x^2) = 0$$

$$(D_x \pm 2u)(u_t + u_{xxx} - 6u^2 u_x) = 0$$

System $Q_{1,4}$ can be reduced to one scalar 6-point equation

$$(u_{0,1}S_1 - u_{2,0})(Q) = 0, \qquad Q = u_{1,0}u_{0,1}(u_{0,0} + u_{1,1}) + 1$$

where $u_{i,j} = S_1^i S_4^j(u)$. (Similar reduction exist for $Q_{2,4}$ and $Q_{3,4}$).
This 6-point equation admits the following local symmetry,

$$\partial_s u_{0,0} = u_{0,0}(S_1 - 1)\frac{1}{(u_{1,0}u_{0,0}u_{-1,0} - 1)(u_{0,0}u_{-1,0}u_{-2,0} - 1)}$$

and a non-local symmetry

$$\partial_x u_{0,0} = u_{0,0}\phi_{0,0}$$

$$(S_1 + 1 + S_1^{-1})(\phi_{0,0}) = (S_1 - 1)\left(u_{0,0}u_{-1,0} + \frac{1}{u_{-1,0}} + \frac{1}{u_{0,0}}\right)$$

$$(S_1^{-1} - S_4)(\phi_{0,0}) = (S_4 - 1)\left(\frac{1}{u_{-1,0}} - \frac{1}{u_{0,0}}\right).$$

Potentiations and 6-point scalar equations

Systems $Q_{2,3}$ and $Q_{1,2}$ can be brought via potentiation and invertible transformations to a 6-point scalar equation,

$$w_{0,1}w_{1,0}w_{1,2}-w_{1,0}w_{2,2}w_{1,2}-w_{0,1}w_{1,0}w_{2,1}-w_{0,0}w_{0,1}w_{2,2}+w_{0,0}w_{1,0}w_{2,2}+w_{0,1}w_{2,1}w_{2,2}=0$$

$Q_{1,3}$ can be brought via potentiation to the equation

$$w_{0,0}w_{1,0}w_{1,2}+w_{0,1}w_{2,1}w_{1,2}-w_{1,0}w_{2,1}w_{1,2}-w_{0,0}w_{2,2}w_{1,2}-w_{0,0}w_{0,1}w_{2,1}+w_{0,0}w_{2,1}w_{2,2}=0$$

Indeed, we introduce w such that $u_{n,m}=\frac{w_{n,m+1}}{w_{n,m}}$ and $v_{n,m}=\frac{w_{n,m}}{w_{n+1,m+1}}$. Here $a_{n,m}$ corresponds to a shifted by n units in the 1 direction and m units in the 3 direction.

These equations seem to be new, they are different from the type of equations classified by V.Adler.

Let **A** be the algebra of functions of the variables $u^j_{p,q}$ for all $p, q \in \mathbb{Z}$ and $j = 1, \ldots, s$.

Each function $f \in \mathbf{A}$ depends on a finite number of the variables $u^j_{p,q}$. Depending on the problem, one can consider polynomial, rational, analytic or meromorphic functions.

There are two commuting automorphisms $\mathcal{S}$ and $\mathcal{T}$ of **A**

$$\mathcal{S}^n\mathcal{T}^m f(u_{i,j}, u_{p,k}, \ldots) = f(u_{i+n,j+m}, u_{p+n,k+m}, \ldots),$$

and thus **A** is a difference algebra.

We consider a system of difference equations of arbitrary order

$$Q^i(u^j_{0,0}, u^j_{1,0}, u^j_{0,1}, u^j_{1,1}, \ldots) = 0, \qquad i = 1, \ldots, r, \qquad j = 1, \ldots, s.$$

In this system we have r equations for s functions $u^1, \ldots, u^s$.
As usual, we assume that equations are valid at every point $(n, m) \in \mathbb{Z}^2$, and thus

$$Q^i_{p,q} = Q^i(u^j_{p,q}, u^j_{p+1,q}, u^j_{p,q+1}, u^j_{p+1,q+1}, \ldots) = 0, \qquad (p, q) \in \mathbb{Z}^2, \qquad i = 1, \ldots, r,$$

With this system we associate the ideal $J_Q = \langle \{Q^i_{p,q}\} \rangle \subset \mathbf{A}$ and the quotient algebra $\mathcal{A} = \mathbf{A}/J_Q$.

For any $a \in J_Q$ one has $\mathcal{S}(a) \in J_Q$ and $\mathcal{T}(a) \in J_Q$. Therefore, $\mathcal{S}$ and $\mathcal{T}$ determine well-defined maps from $\mathcal{A}$ to $\mathcal{A}$ which are automorphisms of $\mathcal{A}$.

Recall that $K = (K^1, \dots, K^s)$ is a *symmetry* of system $Q = 0$ if $Q_*(K) = 0$ modulo J_Q. Here $K^1, \dots, K^s \in \mathbf{A}$ and Q_* is the Frechét derivative of Q

A vector field

$$X_K = \sum_{\substack{j=1,\dots,s,\\ (p,q)\in\mathbb{Z}^2}} \mathcal{S}^p \mathcal{T}^q (K^j) \frac{\partial}{\partial u^j_{p,q}}, \tag{1}$$

is a derivation of the algebra $\mathbf{A}$ and satisfies $\mathcal{S}X_K = X_K\mathcal{S}$, $\mathcal{T}X_K = X_K\mathcal{T}$.

The equation $Q_*(K) = 0$ modulo J_Q implies that $X_K(J_Q) \subset J_Q$. Therefore, X_K determines a well-defined derivation of the algebra $\mathcal{A} = \mathbf{A}/J_Q$.

A (local) symmetry K of the system $Q = 0$ is a map $X_K\colon \mathcal{A} \to \mathcal{A}$ satisfying

$$\mathcal{S}X_K = X_K\mathcal{S}, \qquad \mathcal{T}X_K = X_K\mathcal{T}, \qquad X_K(fg) = fX_K(g)+gX_K(f), \qquad \forall\, f, g \in \mathcal{A}.$$

A *difference extension* of $(\mathcal{A}, \mathcal{S}, \mathcal{T})$ is given by $(\tilde{\mathcal{A}}, \tilde{\mathcal{S}}, \tilde{\mathcal{T}})$, where $\tilde{\mathcal{A}}$ is a commutative associative algebra and $\tilde{\mathcal{S}}, \tilde{\mathcal{T}}$ are automorphisms of $\tilde{\mathcal{A}}$ such that

- the algebra $\mathcal{A}$ is embedded in $\tilde{\mathcal{A}}$,
- the restrictions of $\tilde{\mathcal{S}}$ and $\tilde{\mathcal{T}}$ to $\mathcal{A} \subset \tilde{\mathcal{A}}$ coincide with $\mathcal{S}$ and $\mathcal{T}$ respectively,
- one has $\tilde{\mathcal{S}}\tilde{\mathcal{T}} = \tilde{\mathcal{T}}\tilde{\mathcal{S}}$.

Since $\mathbf{A} \subset \tilde{\mathbf{A}}$ and $J_Q \subset \mathbf{A}$, one has $J_Q \subset \tilde{\mathbf{A}}$. Let $\tilde{\mathcal{J}} \subset \tilde{\mathbf{A}}$ be the ideal generated by $J_Q \subset \tilde{\mathbf{A}}$. Then one has the natural embedding $\mathcal{A} = \mathbf{A}/J_Q \hookrightarrow \tilde{\mathcal{A}} = \tilde{\mathbf{A}}/\tilde{\mathcal{J}}$.

A **nonlocal symmetry** of the system $Q = 0$ in the difference extension $(\tilde{\mathcal{A}}, \tilde{\mathcal{S}}, \tilde{\mathcal{T}})$ of $(\mathcal{A}, \mathcal{S}, \mathcal{T})$ is a map $X\colon \mathcal{A} \to \tilde{\mathcal{A}}$ obeying

$$\tilde{\mathcal{S}}X = X\mathcal{S}, \qquad \tilde{\mathcal{T}}X = X\mathcal{T}, \qquad X(fg) = fX(g) + gX(f) \qquad \forall\, f, g \in \mathcal{A}.$$

Here we use the fact that for any $a \in \mathcal{A}$ and $b \in \tilde{\mathcal{A}}$ the product $ab \in \tilde{\mathcal{A}}$ is well defined, because $\mathcal{A} \subset \tilde{\mathcal{A}}$.

Non-local symmetries

Example: Consider the H1 equation

$$Q = (u_{0,0} - u_{1,1})(u_{1,0} - u_{0,1}) - \alpha + \beta = 0.$$

The following system is compatible modulo J_Q

$$w + w_{1,0} = (u_{0,0} - u_{1,0})^2 + \alpha, \qquad w + w_{0,1} = (u_{0,0} - u_{0,1})^2 + \beta.$$

We can extend $\mathcal{S}, \mathcal{T}$ to new variable w by

$$\begin{aligned} w_{1,0} &= -w + (u_{0,0} - u_{1,0})^2 + \alpha, & w_{0,1} &= -w + (u_{0,0} - u_{0,1})^2 + \beta, \\ w_{-1,0} &= -w + (u_{-1,0} - u_{0,0})^2 + \alpha, & w_{0,-1} &= -w + (u_{0,-1} - u_{0,0})^2 + \beta, \end{aligned}$$

It is easy to check that $K = w$ satisfies $Q_*(K) = 0$ in $\tilde{\mathcal{A}}$ modulo $\tilde{J}$ and, therefore, determines a nonlocal symmetry for H1.

Similar to local symmetries, one can use non-local symmetries to find invariant solutions.

Let us describe solutions of H1 which are invariant with respect to the nonlocal symmetry $K = w$. According to the definition of symmetry-invariant solutions, we need to solve the system

$$Q = (u_{0,0} - u_{1,1})(u_{1,0} - u_{0,1}) - \alpha + \beta = 0,$$

$$w_{1,0} = -w + (u_{0,0} - u_{1,0})^2 + \alpha, \qquad w_{0,1} = -w + (u_{0,0} - u_{0,1})^2 + \beta,$$

$$w_{-1,0} = -w + (u_{-1,0} - u_{0,0})^2 + \alpha, \qquad w_{0,-1} = -w + (u_{0,-1} - u_{0,0})^2 + \beta,$$

$$w = 0.$$

Taking into account $w = 0$, from this system we get

$$u_{1,0} = u_{0,0} + \sqrt{-\alpha}, \qquad u_{0,1} = u_{0,0} + \sqrt{-\beta},$$

which implies that $u(n, m) = n\sqrt{-\alpha} + m\sqrt{-\beta} + c$, where c is a constant.

Happy Burthday!