Nonlinear dynamics of the cosmological scalar fields with singular potentials

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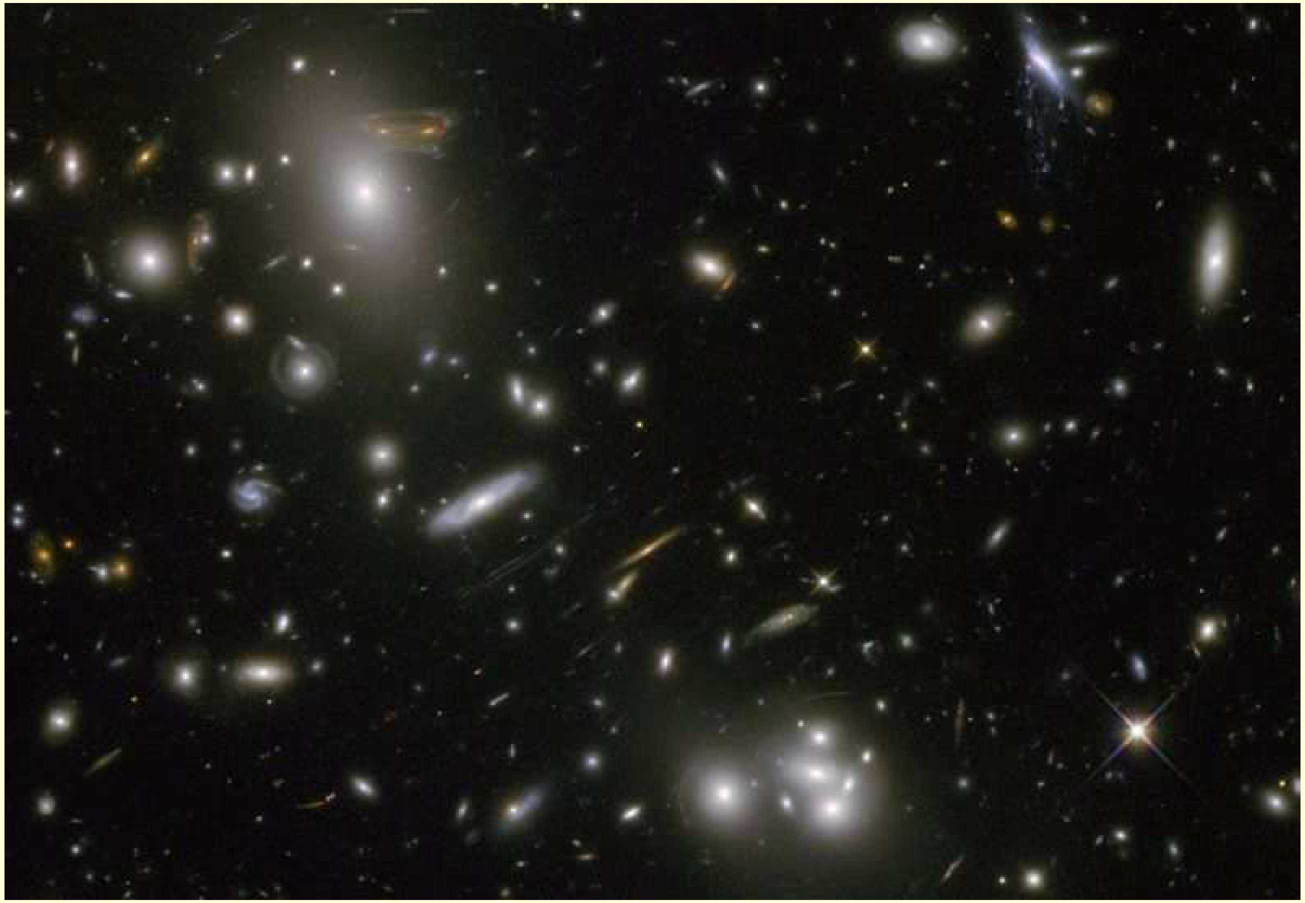


1. Scalar fields in the Friedmann-Robertson-Walker Universe

Consider a scalar field $\phi(t, \mathbf{r})$ governed by the nonlinear KG equation

$$\phi_{tt} + 3H\phi_t - a^{-2}\Delta\phi + U'(\phi) = 0,$$

where $H = a_t/a$ – Hubble parameter, $a(t)$ – scale factor of the FRW metric

$$ds^2 = dt^2 - a^2(t)[dr^2 + r^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2)].$$

Homogeneous inflaton scalar field $\phi(t)$ satisfies

$$\phi_{tt} + 3H\phi_t + U'(\phi) = 0.$$

Friedmann equations

$$\frac{a_{tt}}{a} = -\frac{4\pi G}{3}(\rho + 3p), \qquad \left(\frac{a_t}{a}\right)^2 = \frac{8\pi G}{3}\rho,$$

where

$$\rho = \phi_t^2/2 + U(\phi), \qquad p = \phi_t^2/2 - U(\phi).$$

Continuity equation

$$\rho_t = -3H(\rho + p).$$

2. Equation of state and inflation

In general the Friedmann equations must be supplemented by

$$p = p(\rho)$$

From the equation

$$\frac{a_{tt}}{a} = -\frac{4\pi G}{3}(\rho + 3p)$$

it follows that

$$\text{acceleration, if} \quad w = \frac{p(\rho)}{\rho} < -\frac{1}{3}$$

$$\text{deceleration, if} \quad w = \frac{p(\rho)}{\rho} > -\frac{1}{3}$$

Inflation caused by a scalar field

$$-1 < w = \frac{\phi_t^2/2 - U(\phi)}{\phi_t^2/2 + U(\phi)} \leqslant 1 \quad \text{for any} \quad U(\phi) \geqslant 0$$

a) Slow-roll inflation (Albrecht & Steinhardt 1982, Linde 1982, 1983)

$$\phi_t^2 \ll U(\phi), \quad w \approx -1$$

b) Oscillation-driven inflation (Turner 1983, Damour & Mukhanov 1998)

$$\overline{\phi_t^2} \sim \overline{U(\phi)}$$

3. Dynamics of the scalar field oscillations

The field equation can be written as

$$\phi_{tt} + 2\sqrt{3\pi G}(\phi_t^2 + 2U(\phi))^{1/2}\phi_t + U'(\phi) = 0.$$

Transition to the new independent variables $(\phi, \phi_t) \rightarrow (\rho, \theta)$:

$$\begin{aligned} \phi &= \varphi(\rho, \theta), \\ \phi_t &= \omega(\rho)\varphi_\theta(\rho, \theta), \end{aligned}$$

where $\varphi(\rho, \theta)$ is a 2π-periodic solution of the equation

$$\frac{1}{2}\omega^2(\rho)\varphi_\theta^2 + U(\varphi) = \rho,$$

and the function $\omega(\rho)$ is given by

$$\omega^{-1}(\rho) = \frac{1}{\pi\sqrt{2}} \int_{\varphi_{\min}(\rho)}^{\varphi_{\max}(\rho)} \frac{d\varphi}{\sqrt{\rho - U(\varphi)}}.$$

Compatibility condition is

$$\varphi_\rho \rho_t + \varphi_\theta \theta_t = \omega \varphi_\theta.$$

As a result we obtain

$$\begin{aligned} \rho_t &= -2\sqrt{6\pi G}\rho^{1/2}\omega^2(\rho)\varphi_\theta^2, \\ \theta_t &= \omega(\rho) + 2\sqrt{6\pi G}\rho^{1/2}\omega^2(\rho)\varphi_\rho\varphi_\theta. \end{aligned}$$

Averaging $\rho + p = \omega^2\varphi_\theta^2$ over $(0, 2\pi)$ gives the Turner's formula

$$w(\rho) = \overline{p}/\rho = \gamma(\rho) - 1,$$

$$\gamma(\rho) = \frac{\sqrt{2}\omega(\rho)}{\pi\rho}\int_{\varphi_{\min}(\rho)}^{\varphi_{\max}(\rho)} \sqrt{\rho - U(\varphi)}\, d\varphi.$$

All above formulas are exact ones.

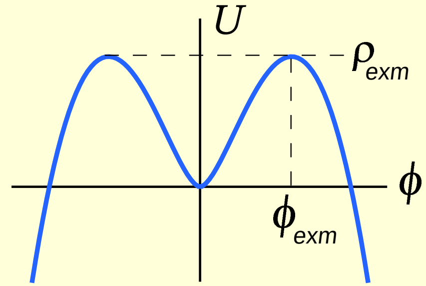
For $\varepsilon \sim H/\omega \ll 1$ the generalized averaging method can be used, in which $\rho, \theta \to \overline{\rho}, \overline{\theta}$. In the lowest order $\rho = \overline{\rho} + O(\varepsilon)$, $\theta = \overline{\theta} + O(\varepsilon)$, that corresponds to the Van der Pol approximation:

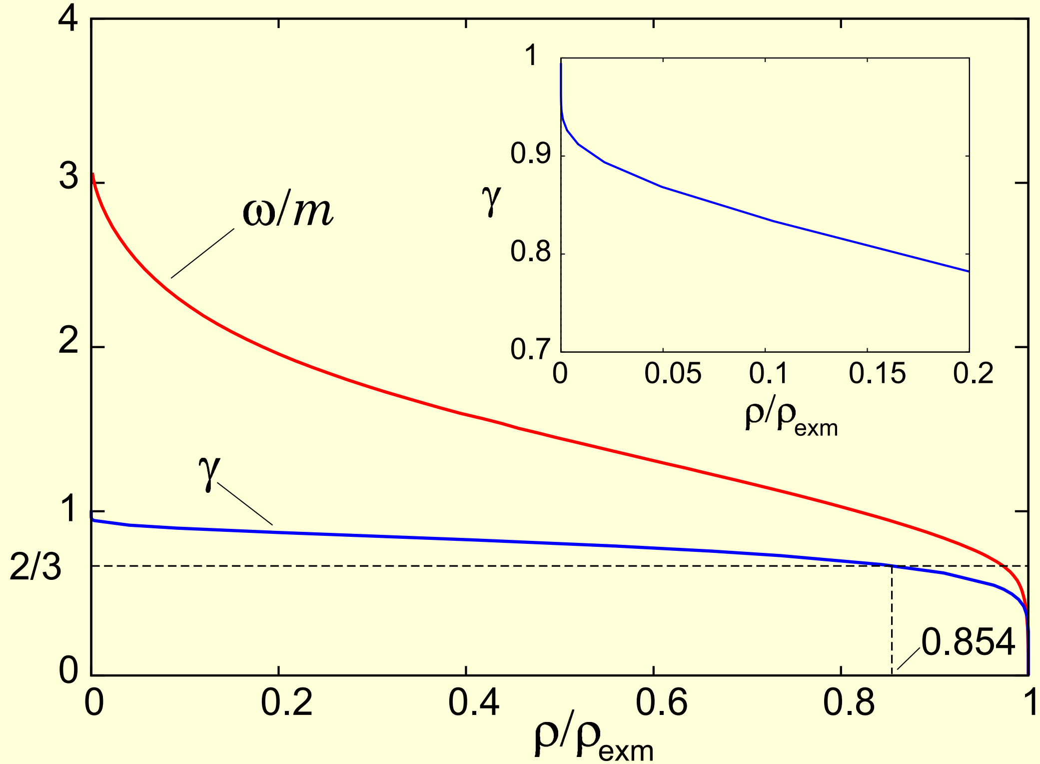
$$\begin{aligned} \rho_t &= -3H\rho\gamma(\rho), \\ \theta_t &= \omega(\rho), \end{aligned}$$

where $H/\omega \sim \varepsilon \ll 1$.

Logarithmic potential

$$U(\phi) = \frac{m^2}{2}\phi^2\left(1 - \ln\frac{\phi^2}{\sigma^2}\right)$$



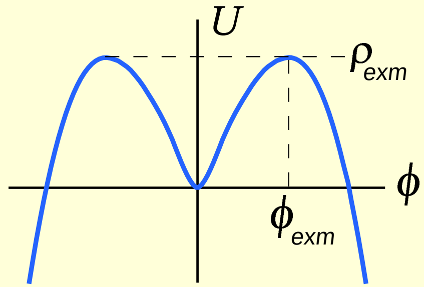


$$\omega/m \simeq \Omega\left(1 - 0.31\,\Omega^{-2} - 0.09\,\Omega^{-4} + O(\Omega^{-6})\right),$$

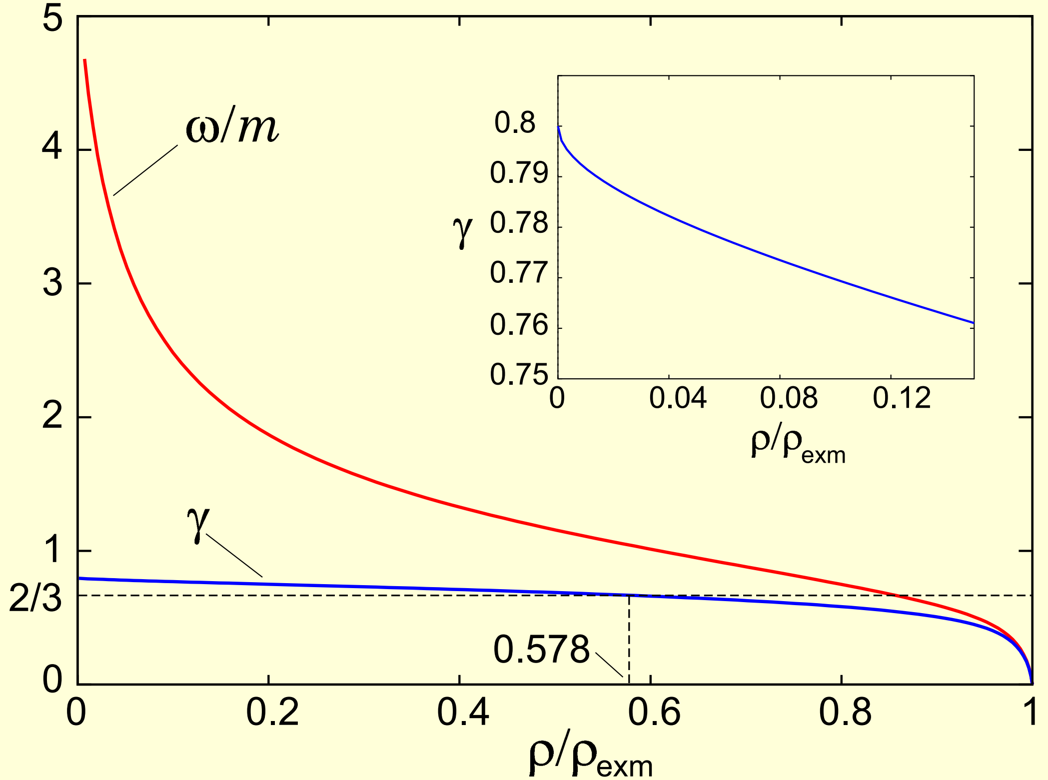
$$\gamma \simeq 1 - 0.5\,\Omega^{-2} - 0.06\,\Omega^{-4} + O(\Omega^{-6}) \qquad (\rho/\rho_{\rm exm} \ll 1),$$

$$\rho/\rho_{\rm exm} = \Omega^2\exp(1-\Omega^2),\quad \Omega^2 = 1 - \ln(\phi_{\rm max}/\phi_{\rm exm})^2,\quad \rho_{\rm exm} = m^2\sigma^2/2,\quad \phi_{\rm exm} = \sigma.$$

Fractional-power potential



$$U(\phi) = -\frac{m^2}{2}\phi^2 + \frac{3\lambda}{4}\phi^{4/3}$$



$$\begin{aligned}\omega/m &\simeq 1.07\,\delta^{-1/3}(1-0.44\,\delta^{2/3}-0.1\,\delta^{4/3}+\mathrm{O}(\delta^2)),\\ \gamma &\simeq (4/5)(1-0.17\,\delta^{2/3}-0.12\,\delta^{4/3}+\mathrm{O}(\delta^2)) \qquad (\rho/\rho_{\rm exm}\ll 1),\end{aligned}$$

$$\rho/\rho_{\rm exm} = -2\delta^2 + 3\delta^{4/3},\quad \delta = \phi_{\max}/\phi_{\rm exm},\quad \rho_{\rm exm} = \lambda^3/(4m^4),\quad \phi_{\rm exm} = \lambda^{3/2}/m^3.$$

4. Instability of the scalar field oscillations

$$\phi_{tt} + 3H\phi_t - a^{-2}\Delta\phi + U'(\phi) = 0.$$

Consider small perturbations around spatially uniform oscillations,

$$\phi(t,\mathbf{r}) = \phi(t) + \delta\phi(t,\mathbf{r}), \quad |\delta\phi| \ll |\phi|, \quad \delta\phi(t,\mathbf{r}) = \int A(t,\mathbf{k}) e^{i\mathbf{kr}} d\mathbf{k},$$

$$A_{tt} + 3HA_t + \left[(k/a)^2 + U''(\phi(t))\right] A = 0.$$

Setting $A = a^{-3/2}\omega^{-1/2}Y(\theta,\mathbf{k})$, $\phi(t) = \varphi(\rho,\theta)$, $\theta_t = \omega(\rho)$, $\rho_t \propto H/\omega \sim \varepsilon$, we arrive at the Hill equation with a slowly varying parameter ρ,

$$d^2Y/d\theta^2 + h(\rho,\theta)Y = 0,$$

$$h(\rho,\theta) = \omega^{-2}(\rho)\left[(k/a)^2 + U''(\varphi(\rho,\theta))\right].$$

Here the function $\varphi(\rho,\theta)$ is 2π-periodic solution of the equation

$$\omega^2(\rho)(\partial^2\varphi/\partial\theta^2) + U'(\varphi) = 0,$$

the dependence $a(\rho)$ and slow evolution of $\rho(\theta)$ are determined by

$$a_\rho/a = -(3\rho\gamma(\rho))^{-1}, \qquad \rho_\theta = -3(H/\omega)\rho\gamma(\rho).$$

We seek resonant solution in the form

$$Y(\theta) = \psi(\rho,\theta)e^{\xi(\theta)},$$

$$\psi = \psi_0(\rho,\theta)+\varepsilon\psi_1(\rho,\theta)+\varepsilon^2\psi_2(\rho,\theta)+..., \quad \xi_\theta = \mathcal{M}_0(\rho)+\varepsilon\mathcal{M}_1(\rho)+\varepsilon^2\mathcal{M}_2(\rho)+...,$$

where $\psi_n(\rho,\theta+2\pi) = \psi_n(\rho,\theta)$. Substitution gives

$$\left[(\partial/\partial\theta \pm \mathcal{M}_0(\rho))^2 + h(\rho,\theta)\right]\psi_n(\rho,\theta) = F_n(\rho,\theta),$$

where $F_0 = 0$, and the functions F_n with $n \geqslant 1$ are 2π-periodic in θ. Setting $\psi_n(\rho,\theta) = X_n(\rho,\theta)e^{-\mathcal{M}_0(\rho)\theta}$ we arrive at

$$\partial^2 X_n/\partial\theta^2 + h(\rho,\theta)X_n = F_n(\rho,\theta)e^{\mathcal{M}_0(\rho)\theta},$$

where ρ is just a parameter. With $n = 0$ we have the standard Hill equation

$$\partial^2 X_0/\partial\theta^2 + h(\rho,\theta)X_0 = 0,$$

so that $\mathcal{M}_0(\rho)$ is the Floquet exponent. Thus, in the lowest approximation

$$Y(\theta) \approx \psi_0(\rho(\theta),\theta)\, e^{\int(\mathcal{M}_0+\varepsilon\mathcal{M}_1)\,d\theta} = \psi_0(\rho(t),\theta(t))\, e^{\int(\mu+\Delta\mu)\,dt},$$

where $\mu = \mathcal{M}_0(\rho)\omega(\rho)$, $\Delta\mu = \varepsilon\mathcal{M}_1(\rho)\omega(\rho) = \frac{3}{4}H\rho\gamma(\rho)\frac{d}{d\rho}\ln W_0^2$.

5. Singular Hill equation and generalized Lindemann-Stieltjes method.

Thus we should consider the Hill equation of the form

$$X_{tt} + h(z(t))X = 0,$$

where $z(t)$ is unbounded τ-periodic function satisfying $z_t^2 = g(z)$.

$$X^+(t) = \psi(t)e^{\mu t}, \quad X^-(t) = \psi(-t)e^{-\mu t}$$

<u>The method</u> Consider $z(t)$ as a new "time" variable, $t \to z(t)$. Then $X^{\pm}(t) = y_{1,2}(z)$, and $w(z) = y_1 y_2 = X^+ X^- = \psi(t)\psi(-t)$ satisfies

$$g(z)w''' + (3/2)g'(z)w'' + [(1/2)g''(z) + 4h(z)]w' + 2h'(z)w = 0.$$

Let $\zeta \leqslant z(t) < \infty$, i.e, the turning points are $\zeta_1 = \zeta$, $\zeta_2 = \infty$.Then the boundary conditions are

$$w(\zeta) = 1,\ w'(\zeta) = w_1,\ w''(\zeta) = -\frac{4h'(\zeta) + [g''(\zeta) + 8h(\zeta)]w_1}{3g'(\zeta)};\ (w'\sqrt{g})_{z\to\infty} \to 0.$$

With $w(z)$ the Floquet exponent μ is given by

$$\mu = \left|\frac{W}{\tau}\int_{\zeta}^{\infty}\frac{dz}{w\sqrt{g}}\right|, \quad W^2 = -4h(\zeta) - g'(\zeta)w_1 > 0.$$

Logarithmic potential

$$U(\phi) = \frac{m^2}{2}\phi^2\left(1 - \ln\frac{\phi^2}{\sigma^2}\right)$$

In this case, after rescaling $\theta/\omega \to t/m$, we set

$$z(t) = -\ln(\phi/\phi_{\max})^2, \quad h(z) = m^{-2}(k/a)^2 - 3 + \Omega^2 + z, \quad g(z) = 4[\Omega^2(e^z - 1) - z],$$

where $\Omega^2 = 1 - \ln(\phi_{\max}/\sigma)^2$. Since $0 \leqslant z < \infty$, the turning points are $\zeta_1 = 0$, $\zeta_2 = \infty$.

The function $w(z)$ satisfies

$$2[\Omega^2(e^z-1)-z]w''' + 3(\Omega^2 e^z - 1)w'' + [\Omega^2(e^z+2) + 2z + 2m^{-2}(k/a)^2 - 6]w' + w = 0,$$

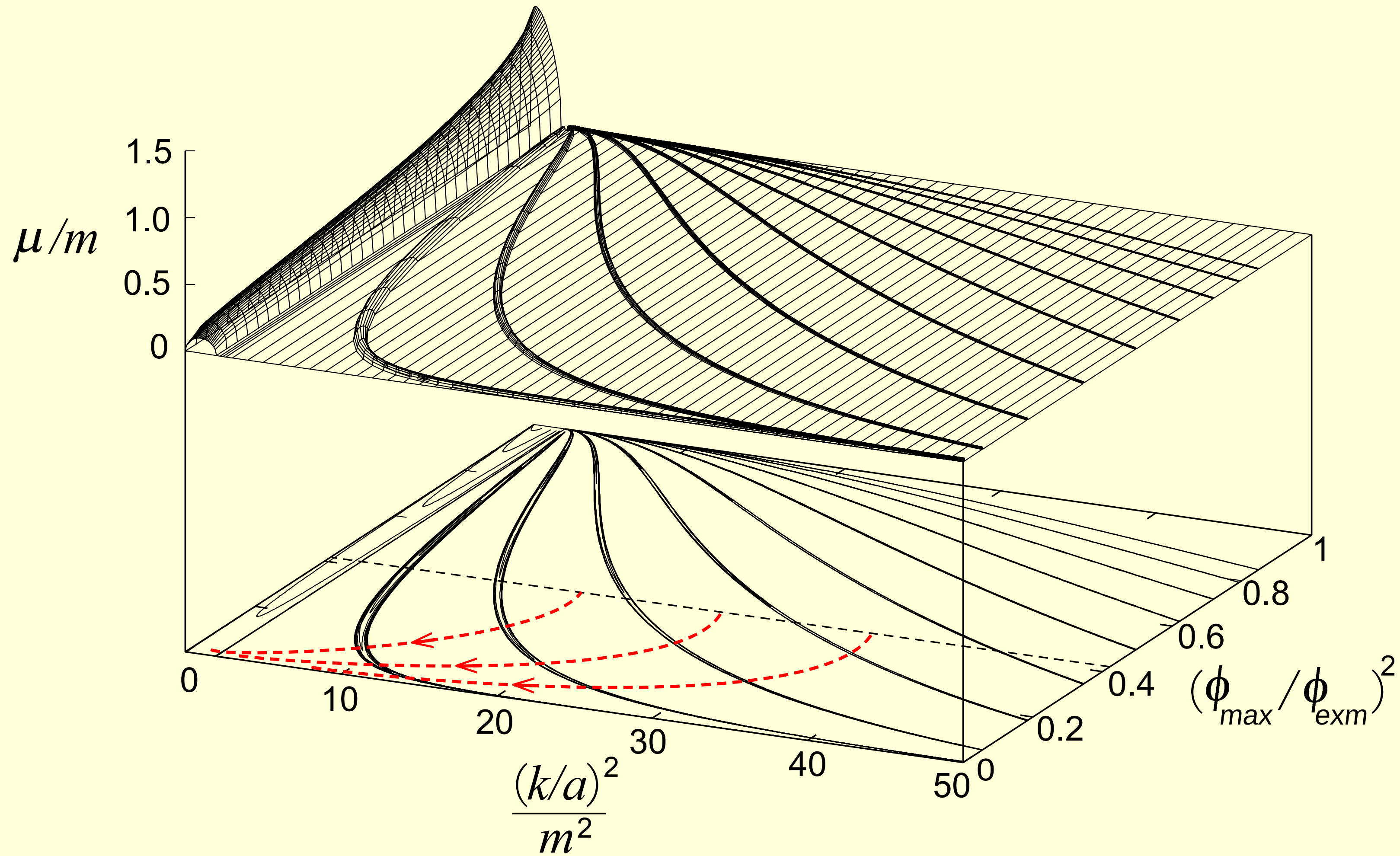
with the boundary conditions

$$w(0) = 1,\ w'(0) = w_1,\ w''(0) = -\frac{1 + [2m^{-2}(k/a)^2 + 3\Omega^2 - 6]w_1}{3(\Omega^2 - 1)};\ (w'\sqrt{g})_{z\to\infty} \to 0.$$

Floquet exponent is μ/m, $\quad \mathcal{M}_0 = \mu/\omega$,

$$\mu\left(\frac{(k/a)^2}{m^2}, \left(\frac{\phi_{\max}}{\phi_{\text{exm}}}\right)^2\right) = \frac{\omega}{\pi}\left|W\int_0^\infty \frac{dz}{w\sqrt{g}}\right|,\ W^2 = 4\left[3 - \Omega^2 - \frac{(k/a)^2}{m^2} - (\Omega^2 - 1)w_1\right].$$

Stability-instability chart



$$\left(\frac{k}{a}\right)^2 \underset{t\to\infty}{\simeq} const \left(\frac{\phi_{\rm max}}{\phi_{\rm exm}}\right)^{4/3}, \quad A(t,\mathbf{k}) \sim a^{-3/2}\omega^{-1/2} \exp\int (\mu+\Delta\mu)\, dt.$$

$$\phi_{tt} + 3H\phi_t - a^{-2}\Delta\phi + U'(\phi) = 0,$$

$$U(\phi) = \frac{m^2}{2}\phi^2\left(1 - \ln\frac{\phi^2}{\sigma^2}\right).$$

Not counting the cosmological expansion, $H \approx 0$, $a \approx const$, the equation

$$\phi_{tt} - a^{-2}\Delta\phi - m^2\phi\ln(\phi/\sigma)^2 = 0$$

has the exact pulson solution

$$\phi(t,r) = \Phi(t)e^{-(r/r_0)^2},$$

that is periodic in time and localized in space with the characteristic scale $r_0 = \sqrt{2}(am)^{-1}$. The corresponding wave number $k \sim am$, just right to have

$$\frac{(k/a)^2}{m^2} \sim 1.$$

The oscillating field decays into the pulsons!

Fractional-power potential

$$U(\phi) = -\frac{m^2}{2}\phi^2 + \frac{3\lambda}{4}\phi^{4/3}$$

In this case, after rescaling $\theta/\omega \to t/m$,

$$\begin{aligned} z(t) &= (\phi/\phi_{\max})^{-2/3}, \quad h(z) = m^{-2}(k/a)^2 - 1 + (\beta/3)z, \\ g(z) &= (4/9)z^2(z-1)\left[(3\beta/2-1)\,z^2 + (3\beta/2-1)\,z - 1\right], \end{aligned}$$

where $\beta = (\phi_{\max}/\phi_{\text{exm}})^{-2/3} > 1$. Turning points: $\zeta_1 = 1$, $\zeta_2 = \infty$.
The function $w(z)$ satisfies

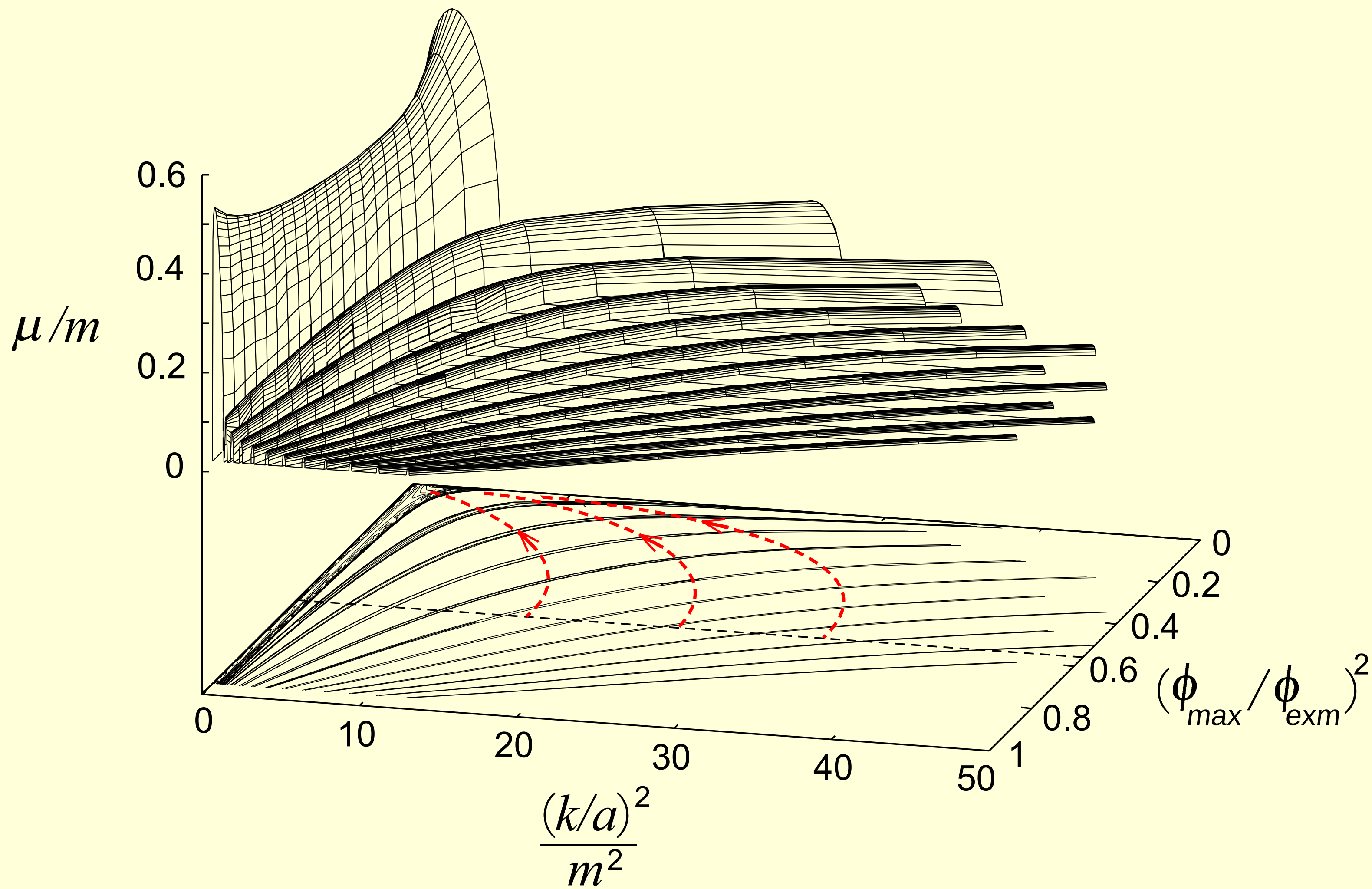
$$2z^2(z-1)[(3\beta-2)(z+1)z-2]w''' + 3z[5(3\beta-2)z^3 - 9\beta z + 4]w''$$
$$+[20(3\beta-2)z^3 - 6\beta z + 36m^{-2}(k/a)^2 - 32]w' + 6\beta w = 0,$$

$$w(1) = 1,\ w'(1) = w_1,\ w''(1) = -\frac{\beta/3 + [2m^{-2}(k/a)^2 + 3\beta - 4]w_1}{\beta - 1};\ (w'\sqrt{g})_{z\to\infty} \to 0.$$

As a result, $\mathcal{M}_0 = \mu/\omega$,

$$\mu\left(\frac{(k/a)^2}{m^2}, \left(\frac{\phi_{\max}}{\phi_{\text{exm}}}\right)^2\right) = \frac{\omega}{\pi}\left|W\int_1^\infty \frac{dz}{w\sqrt{g}}\right|,\ W^2 = \frac{4}{3}\left[3 - \beta - 3\frac{(k/a)^2}{m^2} - (\beta-1)w_1\right].$$

Stability-instability chart



$$\left(\frac{k}{a}\right)^2 \underset{t\to\infty}{\simeq} const \left(\frac{\phi_{\rm max}}{\phi_{\rm exm}}\right)^{10/9}, \quad A(t,\mathbf{k}) \sim a^{-3/2}\omega^{-1/2}\exp\int(\mu+\Delta\mu)\,dt.$$


Thank you for attention!