Polynomial forms for Calogero-type Hamiltonians

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1. Introduction

Consider integrable Hamiltonians

$$H = \Delta + U(x_1, ..., x_n)$$

related to simple Lie algebras. For such Hamiltonians the potential U is a rational, trigonometric or elliptic function. For instance, the elliptic Calogero-Moser Hamiltonian is given by

$$H = \Delta + g \sum_{i>j} \wp(x_i - x_j).$$

Observation 1 (A.Turbiner). Many of these Hamiltonians admit a change of variables that bring it to a differential operator with polynomial coefficients.

Example. Consider the Calogero model with $n = 3$:

$$H = \Delta + g \sum_{i>j}^{3} \frac{1}{(x_i - x_j)^2}.$$

Let $Y = \sum_{i=1}^{3} x_i$ and $y_i = x_i - \frac{Y}{3}$. Then

$$\Delta = -3\frac{\partial^2}{\partial Y^2} - \frac{2}{3}\Big(\frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2} - \frac{\partial^2}{\partial y_1 \partial y_2}\Big).$$

Thus we have reduced the Hamiltonian to the following two dimensional one:

$$\mathcal{H} = -\frac{1}{3}\Big(\frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2} - \frac{\partial^2}{\partial y_1 \partial y_2}\Big) + \nu(\nu - 1) \sum_{i>j}^{3} \frac{1}{(y_i - y_j)^2}. \quad (1)$$

Here $y_3 = -y_1 - y_2$.

The transformation

$$x = -y_1^2 - y_2^2 - y_1 y_2, \qquad y = -y_1 y_2 (y_1 + y_2)$$

brings $\mathcal{H}$ to the polynomial form

$$L = -2x \frac{\partial^2}{\partial x^2} - 6y \frac{\partial^2}{\partial x \partial y} + \frac{2}{3} x^2 \frac{\partial^2}{\partial y^2} - 2(1 + 3\nu) \frac{\partial}{\partial x}. \quad \square$$

In the trigonometric case the transformation to a polynomial form is given by

$$x = \cos y_1 + \cos y_2 + \cos (y_1 + y_2) - 3,$$

$$y = \sin y_1 + \sin y_2 - \sin (y_1 + y_2).$$

Theorem. The transformation

$$x = \frac{\wp'(y_1) - \wp'(y_2)}{\wp(y_1)\wp'(y_2) - \wp(y_2)\wp'(y_1)}, \qquad y = \frac{\wp(y_1) - \wp(y_2)}{\wp(y_1)\wp'(y_2) - \wp(y_2)\wp'(y_1)}$$

brings the elliptic Calogero-Moser Hamiltonian to a polynomial form.

Factorization of the Wronskian.

Consider the Wronkian $W = \frac{\partial y}{\partial y_2}\frac{\partial x}{\partial y_1} - \frac{\partial x}{\partial y_2}\frac{\partial y}{\partial y_1}$ of the transformation. It turns out that W can be written in the factorized form:

$$W(x,y) = \frac{\sigma(x-y)\,\sigma(x+2y)\,\sigma(-y-2x)}{\sigma_1^3(x)\,\sigma_1^3(y)\,\sigma_1^3(-x-y)}.$$

Here σ the Weierstrass sigma-function. The function σ_1 is the sigma-function associated with any half-period ω. By definition,

$$\sigma_1(x) = \frac{\sigma(x+\omega)}{\sigma(\omega)}\,\exp\Big(-\frac{\sigma'(\omega)}{\sigma(\omega)}\,x\Big).$$

Notice that for the trigonometric degeneration we arrive at

$$W(x,y) = \frac{\sin(x-y)\,\sin(x+2y)\,\sin(-y-2x)}{\cos^3(x)\cos^3(y)\cos^3(x+y)}.$$

2. Classification of the polynomial forms

Consider second order differential operators

$$L = a(x,y)\frac{\partial^2}{\partial x^2} + 2b(x,y)\frac{\partial^2}{\partial x \partial y} + c(x,y)\frac{\partial^2}{\partial y^2} + d(x,y)\frac{\partial}{\partial x} + e(x,y)\frac{\partial}{\partial y} + f(x,y) \quad (2)$$

with polynomial coefficients. Denote by $D(x,y)$ the determinant $a(x,y)c(x,y) - b(x,y)^2$.

The operators we are interested in should have three important properties:

Property 1. We assume that the associated contravariant metric

$$g^{1,1} = a\ ,\ g^{1,2} = g^{2,1} = b\ ,\ g^{2,2} = c\ ,$$

is flat.

This is equaivalent to

$$R_{1,2,1,2} = ab^2\, b_{xx} + \cdots + c^2 d\, a_y = 0.$$

Example. For any constant α, β, γ the metric g with

$$a = x^3 - 3xy + \alpha(x^2 - 2y) + \beta x + 2\gamma,$$

$$b = x^2 y - 2y^2 + \alpha xy + 2\beta y + \gamma x,$$
$$c = xy^2 + 2\alpha y^2 + \beta xy + \gamma(x^2 - 2y)$$

is flat.

Property 2. The operator should be potential. This means that

$$\frac{\partial}{\partial y}\Big(\frac{be - cd + c(a_x + b_y) - b(b_x + c_y)}{D}\Big) \tag{3}$$

$$= \frac{\partial}{\partial x}\Big(\frac{bd - ae + a(b_x + c_y) - b(a_x + b_y)}{D}\Big).$$

The properties 1 and 2 guaranty that L can be reduced to the form

$$\bar{L} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + V(x, y)$$

by a proper change of variables.

Observation 2. (A. Turbiner). Known polynomial forms for the Calogero-Moser type Hamiltonians preserve some finite - dimensional vector spaces of polynomials.

In this talk we considier operators (2) with polynomial coefficients that satisfy the following condition:

Property 3. The operator has to preserve the vector space of all polynomials $P(x, y)$ such that $\deg P \leqslant n$ for some $n > 1$.

If L satisfies Property 3 then the coefficients of L have the following structure

$$a = q_1x^4 + q_2x^3y + q_3x^2y^2 + k_1x^3 + k_2x^2y + k_3xy^2 +$$
$$a_1x^2 + a_2xy + a_3y^2 + a_4x + a_5y + a_6;$$

$$b = q_1x^3y + q_2x^2y^2 + q_3xy^3 + \frac{1}{2}\Big(k_4x^3 + (k_1+k_5)x^2y + (k_2+k_6)xy^2 + k_3y^3\Big) +$$
$$b_1x^2 + b_2xy + b_3y^2 + b_4x + b_5y + b_6;$$

$$c = q_1x^2y^2 + q_2xy^3 + q_3y^4 + k_4x^2y + k_5xy^2 + k_6y^3 +$$
$$c_1x^2 + c_2xy + c_3y^2 + c_4x + c_5y + c_6;$$

$$d=(1-n)\Big(2(q_1x^3+q_2x^2y+q_3xy^2)+k_7x^2+(k_2+k_8-k_6)xy+k_3y^2\Big)+$$
$$d_1x+d_2y+d_3;$$
$$e=(1-n)\Big(2(q_1x^2y+q_2xy^2+q_3y^3)+k_4x^2+(k_5+k_7-k_1)xy+k_8y^2\Big)+$$
$$e_1x+e_2y+e_3;$$
$$f=n(n-1)\Big(q_1x^2+q_2xy+q_3y^2+(k_7-k_1)x+(k_8-k_6)y\Big)+f_1.$$

The dimension of the space of such operators equals 36. The group GL_3 acts on this vector space by the formula

$$\tilde{x}=\frac{P}{R},\qquad \tilde{y}=\frac{Q}{R},\qquad \tilde{L}=R^{-n}LR^{n},$$

where P,Q,R are polynomials of degree one in x and y.

This representation is a sum of irreducible representations V_1, V_2 and V_3 of dimensions 27, 8 and 1 correspondingly. A basis in V_2 is given by

$$x_1 = 5k_7 - k_5 - 7k_1, \qquad x_2 = 5k_8 - k_2 - 7k_6,$$

$$x_3 = 5d_1 + 2(n-1)(2a_1 + b_2), \qquad x_4 = 5e_1 + 2(n-1)(2b_1 + c_2),$$

$$x_5 = 5d_2 + 2(n-1)(2b_3 + a_2), \qquad x_6 = 5e_2 + 2(n-1)(2c_3 + b_2),$$

$$x_7 = 5d_3 + 2(n-1)(a_4 + b_5), \qquad x_8 = 5e_3 + 2(n-1)(b_4 + c_5).$$

The generic orbit of the action on V_2 has dimension 6. There are two polynomial invariants of the action:

$$I_1 = x_3^2 - x_3x_6 + x_6^2 + 3x_4x_5 + 3(n-1)(x_1x_7 + x_2x_8),$$

and

$$I_2 = 2x_3^3 - 3x_3^2x_6 - 3x_3x_6^2 + 2x_6^3 + 9x_4x_5(x_3 + x_6) +$$

$$9(n-1)(x_1x_3x_7 + x_2x_6x_8 - 2x_1x_6x_7 - 2x_2x_3x_8 + 3x_2x_4x_7 + 3x_1x_5x_8).$$

Flat potential operators with discrete symmetries

For almost all known examples the operator L that satisfies Properties 1-3 possesses additional finite group of discrete symmetries.

Example. The operator with coefficients

$$a = x^2(x^2 + y^2) + \alpha x^2 + \beta y^2, \qquad b = xy(x^2 + y^2) + (\alpha - \beta)xy,$$

$$c = y^2(x^2 + y^2) + \beta x^2 + \alpha y^2, \qquad d = 2(n-1)x(\lambda - x^2 - y^2),$$

$$e = 2(n-1)y(\lambda - x^2 - y^2), \quad f = n(n-1)(x^2 + y^2).$$

satisfies Properties 1-3, and possesses the discrete group of symmetries isomorphic D_4, generated by reflections

$$x \to -x,\ y \to y, \qquad x \to x,\ y \to -y, \qquad x \to y,\ y \to x.$$

Consider the case when L is invariant with respect to a reflection. Using a transformation, we reduce the reflection to the form $\tilde{x} = x$, $\tilde{y} = -y$. Then the coefficients of the operator L have the following symmetry properties:

$$a(x,-y) = a(x,y), \quad b(x,-y) = -b(x,y), \quad c(x,-y) = c(x,y),$$

$$d(x,-y) = d(x,y), \quad e(x,-y) = -e(x,y), \quad f(x,-y) = f(x,y).$$

The class of such operators admits the transformation group

$$\tilde{x} = \frac{\alpha x + \beta}{\gamma x + \delta}, \qquad \tilde{y} = \frac{y}{\gamma x + \delta}. \tag{4}$$

Transformations $\tilde{L} = c_1 L + c_2$ are also allowed.

The coefficients a, b and c can be written in the form

$$a = P + Qy^2, \qquad b = \frac{1}{4}(P' - R)y + \frac{1}{2}Q'y^3,$$

$$c = \Big(S + \frac{1}{12}P'' - \frac{1}{4}R' + \sigma\Big)y^2 + \frac{1}{2}Q''y^4.$$

where $\deg P = 4, \quad \deg Q = \deg R = \deg S = 2.$

Under transformations (4) the polynomial P changes as follows

$$\tilde{P} = (\gamma x + \delta)^4 P\Big(\frac{\alpha x + \beta}{\gamma x + \delta}\Big). \tag{5}$$

Definition. A differential operator L is called **elliptic** if the polynomial P has four different roots on the Riemann sphere. It is called **trigonometric** if P has one double root.

Classification of the elliptic models

In the elliptic case without loss of generality we set

$$P(x) = x(x-1)(x-u).$$

Proposition 1. If the property 1 holds then any root of the polynomial S is a root of the polynomial P. □

It follows from Proposition 1 that there are two alternatives: **A:** $S = kx^2$ and **B:** $S = kx(x-1)$.

Theorem 1. In Case **A** we obtain from $R_{1,2,1,2} = 0$ that

$$S(x) = x^2, \quad R(x) = -\frac{5}{3}(x^2 - 2x + 3u - 2ux),$$

$$Q(x) = \frac{1}{9}(x^2 - x + 1 + u^2 - ux - u), \qquad \sigma = 0.$$

It follows from (3) that

$$d = \frac{1}{9}(1-n)\left(3(5x^2 - 4x - 4ux + 3u) + (2x - 1 - u)\, y^2\right),$$

$$e = \frac{2}{9}(1-n)\, y\left(9x+y^2-6u-6\right), \qquad f = \frac{1}{9}n(n-1)\left(6x+y^2\right). \qquad \square$$

Theorem 2. In Case **B**

$$S(x) = x(x-1), \quad R(x) = -3(x^2 - 2ux + u),$$

$$Q(x) = \frac{1}{2}(x^2 - 2ux + 2u^2 - u), \qquad \sigma = \frac{1}{3}(2u-1),$$

$$d = (1-n)\Big(2\lambda_1(x-u) + \lambda_2(x^2 - x + 2ux - 4u^2 + 2u) +$$

$$2(1-u)(x-2u) + (x-u)\,y^2\Big),$$

$$e = (1-n)\Big(\lambda_1 y + \lambda_2 xy + xy + y^3\Big),$$

$$f = n(n-1)\Big(\lambda_2 x - x + \frac{1}{2}y^2\Big).$$