Stability of autoresonance under persistent perturbation

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Outline

Summary

A dynamical system under white noise perturbation is considered.

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Theorem of stability of equilibrium is proved.

The method of parabolic equation is used.

An appropriate barrier function for the Kolmogorov's equation is the main mathematical achievement.

The result is applied to prove the stability of the autoresonance phenomenon.

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Unperturbed system is ODEq

$$\frac{d\mathbf{y}}{dT} = \mathbf{a}(\mathbf{y}, T), \quad \mathbf{y} \in \mathbb{R}^n, \quad T > 0. \tag{1}$$

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The problem of stability

Does the trajectory $\mathbf{y} = \mathbf{y}_\mu(T; \mathbf{x})$ remain near equilibrium, if the perturbation $\mu, \mathbf{x}$ are small and the matrix is in a ball $||B|| < M$?

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How to understand the stability under white noise?

There are some Khasminskii's and Freidlin–Wentzell's results concerning either dissipative or autonomous systems.

References

Khasmiskii R.

Stochastic Stability of Differential Equations Series: Stochastic Modelling and Applied Probability, Vol. 66. Springer, NewYork, 1980

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concerns either the dissipative systems or a special perturbation $B(0, T) \equiv 0$.

Р. З. ХАСЬМИНСКИЙ

Устойчивость систем дифференциальных уравнений при случайных возмущениях их параметров

ИЗДАТЕЛЬСТВО «НАУКА»
ГЛАВНАЯ РЕДАКЦИЯ ФИЗИКО-МАТЕМАТИЧЕСКОЙ ЛИТЕРАТУРЫ
МОСКВА 1969

References

Freidlin, M.I., Wentzell, A.D.

Random perturbations of Dynamical Systems. 2nd Edition, Springer, NewYork, 1998

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References

ТЕОРИЯ ВЕРОЯТНОСТЕЙ
И МАТЕМАТИЧЕСКАЯ СТАТИСТИКА

А. Д. ВЕНТЦЕЛЬ, М. И. ФРЕЙДЛИН

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What the problems?

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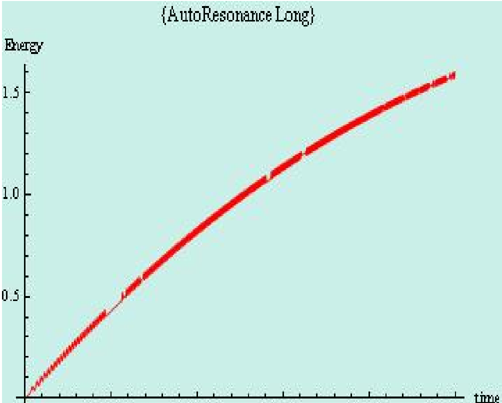

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Autoresonance solutions:

$$\rho(t) \to \infty, t \to \infty.$$

Applications

Model systems of autoresonance:

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Main autoresonance equations

$$\frac{d\rho}{dt} = \sin\Psi, \quad \rho\left[\frac{d\Psi}{dt} - \rho^2 + \lambda t\right] = \cos\Psi.$$

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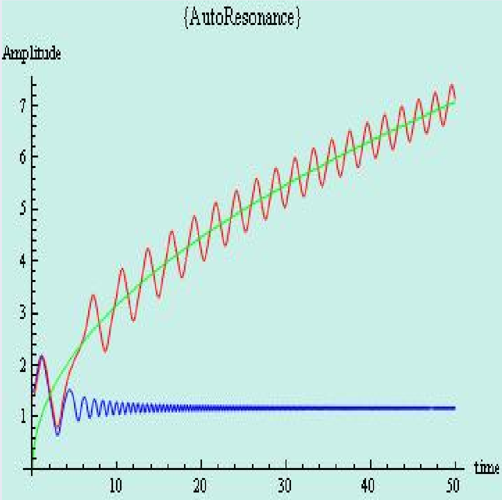
Problem

Are stable these solutions with respect to random perturbations?

Initial stage of autoresonance

Autoresonance solutions



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If there exists a local Lyapunov function then the equilibrium is stable with respect to persistent perturbations.

The result is right for some random perturbations, but not for the white noise (Krasovskii).

И. Г. МАЛКИН

ТЕОРИЯ УСТОЙЧИВОСТИ ДВИЖЕНИЯ

ГОСУДАРСТВЕННОЕ ИЗДАТЕЛЬСТВО
ТЕХНИКО-ТЕОРЕТИЧЕСКОЙ ЛИТЕРАТУРЫ
МОСКВА 1952 ЛЕНИНГРАД

Н. Н. КРАСОВСКИЙ

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Strong stability under white noise can not be at all.

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Weak stability under white noise may be, if the system is *dissipative* (Khasminskii). It is not the autoresonance case.

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Limited weak stability is derived from Freidlin's results if the system is *autonomous*. It is not the autoresonance case.

Stability under given estimates

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$$dy = a(y, T)dT + \mu B(y, T)\, dw(T), \;\; T > 0; \;\; y|_{T=0} = x.$$

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Limited weak stability (Krasovskii) with given estimates:
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The given estimates are more appropriate at physics, especially at the autoresonance phenomenon. It is not the Freidlin's case.

Main result

Perturbed system

$$d\mathbf{y} = \mathbf{a}(\mathbf{y}, T)dT + \mu B(\mathbf{y}, T)\, d\mathbf{w}(T),\ \ T > 0;\ \ \mathbf{y}|_{T=0} = \mathbf{x}.$$

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Let unperturbed deterministic system has a local Lyapunov function $U(\mathbf{y}, T)$ with the properties

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Then the equilibrium is limited weak stable (under white noise) with estimates: $\delta(\varepsilon) = \delta_M\sqrt{\varepsilon},\ \ \Delta(\varepsilon) = \Delta_M\sqrt{\varepsilon},\ \ (\delta_M, \Delta_M = \text{const})$.

Conclusion

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Open problem

Is there weak stability on a very long time interval, for example, $0 < T < \mathcal{O}(\exp(\mu^{-2}))$?

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Autoresonance perturbation of the pendulum

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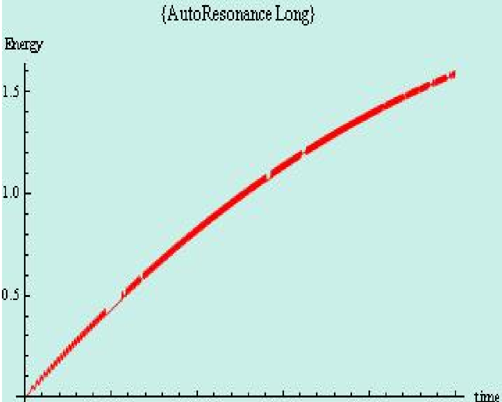

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Thanks

THANK YOU FOR ATTENTION!

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Problem: how to construct an appropriate barrier

$$V(\mathbf{x}, t; T, \mu) = \mathcal{O}(|\mathbf{x}|^2 + \mu^2), \ \mathbf{x} \to 0, \ \mu \to 0.$$

Barrier skill

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Lyapunov function $U(\mathbf{y}, T)$ with the local properties
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Local part of barrier

$$V_1(\mathbf{x}, t; T, \mu) = m\mu^2 \exp\left(\alpha t + \frac{U - \rho}{\mu^2}\right), \ m, \rho = \mathrm{const} > 0$$