Nonlinear dynamics of trapped waves on currents: a new paradigm and novel mechanisms of freak wave formation

Victor Shrira
and
Alexey Slunyaev

Keele University, UK

V.E.Zakharov -75
SCT-2914
4-8 August 2014, Chernogolovka

Outline

Outline

Motivation

Rogue wave on currents

Wave on currents : theoretical background and the fundamental shortcoming of the existing paradigm

The main idea

Problem statement

Linear theory

Boundary-value problem

Dispersion relation

Nonlinear theory

3-wave interactions

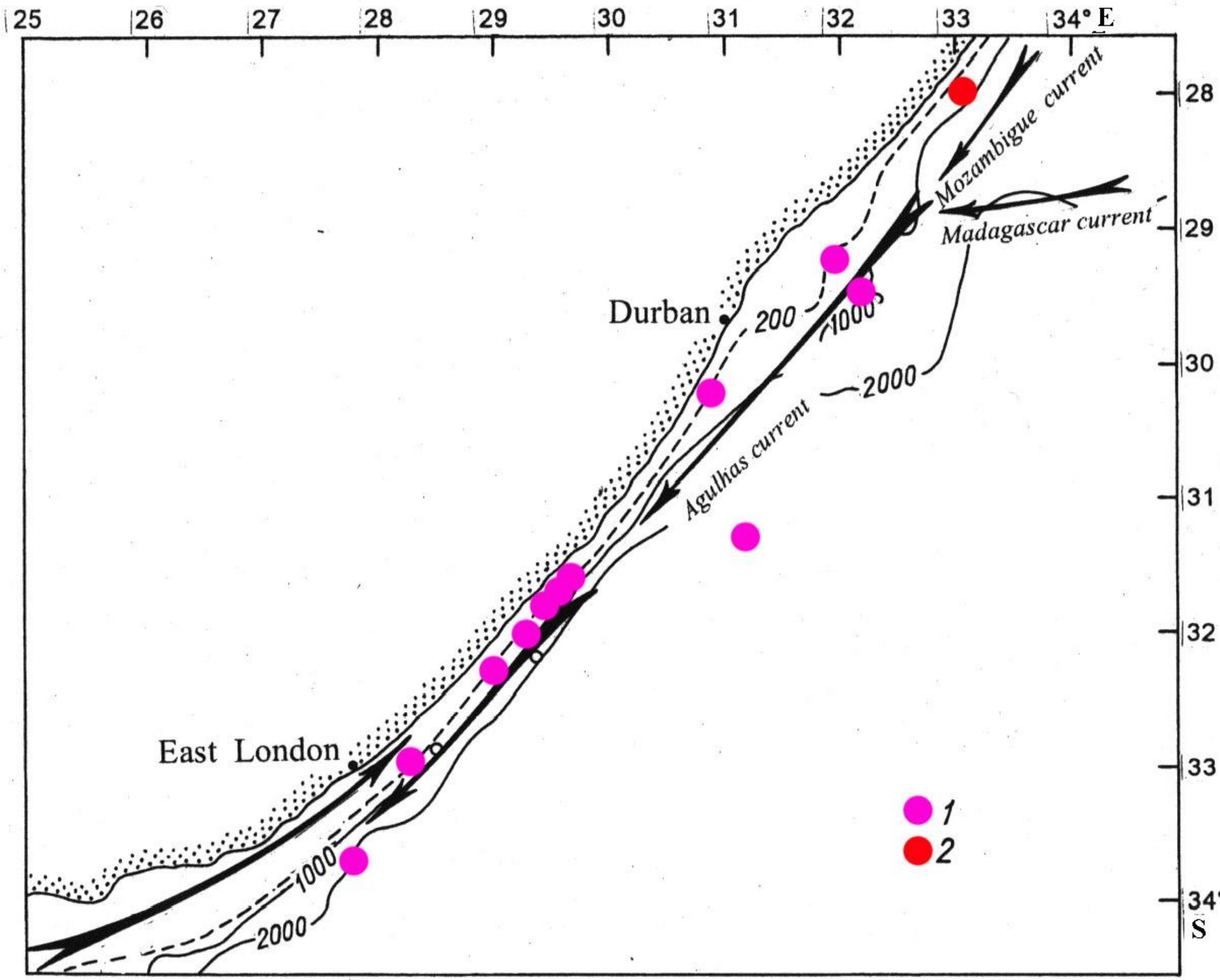
4-wave interactions

Discussion

Wave enhancing effects


South Africa
Madagascar
Maputo
Durban
East London
Cape Town
Agulhas Current

Ship accidents on Agulhas current



Gaastekerk **(Apr'52)**

Oranjefontain **(Sep'53)**

Jagersfontain **(Dec'59)**

Edinburgh Castle **(Aug'64)**

World Glory **(Jun'68)**

Esso Lancashire **(Aug'68)**

Clan Maclay **(Oct'69)**

Southern Cross **(Oct'69)**

Moreton Bay **(Aug'71)**

Bencruachan **(May'73)**

Svealand **(Sep'73)**

Taganrogsky Zaliv **(Apr'85)**

(Suez Canal was closed for 1967-1975, what causeded more intense navigation along the coast of Africa)

[Mallory, 1974; Lavrenov, 1998]

Ship accidents on Agulhas current

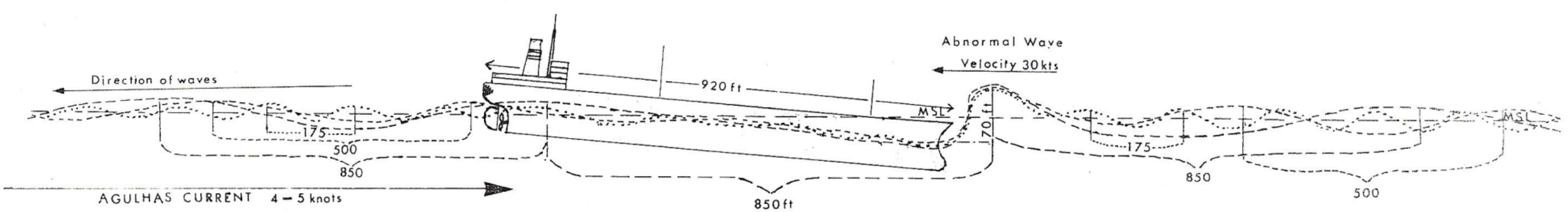
All accidents occurred close to the maximum of the current

There were coexisting different wave systems

Waves were propagating against the current

Physical mechanisms (?)

Not the superposition effect!



[Mallory, ‘74]

Overview of theoretical approaches

- Linear models (*ray theory, caustics)* Peregrine & Smith (CambPhilSoc'75, RoySoc'79), Peregrine (AdvApplMath'76), Smith (JFM'76): *trapped modes, dispersion relations,, nonlinear effects on caustics*
- Lavrenov (NatHaz'98): *rays on a jet current, simulations, Agulhas current conditions*
- White & Fornberg (JFM'98): *statistics for random current fluctuations:*

Moreira & Peregrine (JFM'12), fully nonlinear simulations

NLS models:

Smith JFM1976, Turpin et al, JFM 1983, Gerber JFM 1987, Stocker & Peregrine JFM 1999, Hjelmervik & Trulsen, JFM 2009, Onorato et al, PRL 2011 : current is weak

increase of steepness on longitudinally inhomogeneous opposite currents

triggers BF instability and strong departure from Gaussianity

● DNS: TT Janssen & Herbers (JPO'09): The increase of steepness on opposing current leads to BF instability, etc. Non-NLS features: (i) formation of trapped waves. (ii) Broad spectrum after the increase of kurtosis.

● Observations: Kudryavtsev et al (JGR 95) observed trapped wind waves on the Gulfstream.

The existing theoretical approaches are not suitable for describing nonlinear dynamics of trapped waves. Hence, we know close to nothing.

The state of the art: the existing paradigm has fundamental shortcomings

The existing theoretical approaches are not adequate for describing wave nonlinear interactions in inhomogeneous situations.

(i) It is implicitly assumed that the nonlinear interactions remain the same and just adjust adiabatically. For this to be true the characteristic scales of nonlinear interactions L_{Nl} should be much smaller than the scale of inhomogeneity *L*.

$L_{Nl(d)} \sim \lambda\ \varepsilon^{-2}$

(ii) Refraction occurs in the *x*-space, nonlinear interactions live in the *k*-space.

The main idea

If the jet currents are longitudinally uniform, then the solutions to linearized equations of hydrodynamics for water waves could be always presented in a separable form: as waves propagating along the current with some `modal' dependence on the vertical and transverse variables.

The reason the modal description has not been developed: it **is not easy to find these modes, one has to solve a 2-d BVP.**

Here we found a way to solve this problem asymptotically under some mild assumptions. This finding provides the foundation for systematic weakly nonlinear theory of wave dynamics on jet currents.

The main idea

The trapped modes differ qualitatively from the free waves. This fact profoundly changes all aspects of their nonlinear dynamics

The rogue wave implications

There is a crucial difference in wave dynamics between the 1D evolution and 2D evolution

(Onorato et al (PhysFl'02, PRL'09), Waseda (06), Gramstad & Trulsen (JFM'07), Mori et al (JGR'07),

Annenkov & Shrira GRL09, (JPO14)

Waves trapped by jet currents =>
described in modal representation

Effectively **unidirectional** (nonlinear) evolution

Increase of rogue wave likelihood
due to nonlinear self-modulation effects

Outline

Motivation

Rogue wave on currents
Wave on currents' theories
The main idea

Problem statement

Linear theory

Boundary-value problem and dispersion relation

Nonlinear theory

3-wave interactions (between trapped waves)
4-wave interactions (between trapped waves)
3-wave interactions (between trapped and passing through waves)

Discussion

Wave enhancing effects

Outline

Motivation

- Rogue wave on currents
- Wave on currents theories
- The main idea

Problem statement

Linear theory

- Boundary-value problem
- Dispersion relation

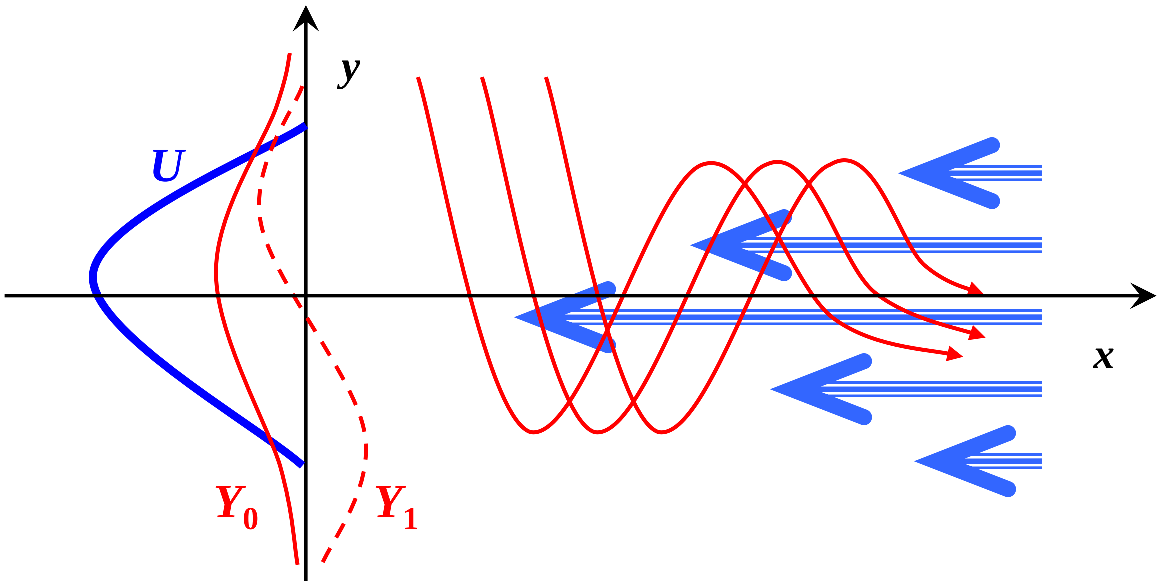
Nonlinear theory

- 3-wave interactions
- 4-wave interactions

Discussion

- Wave enhancing effects

Modal representation



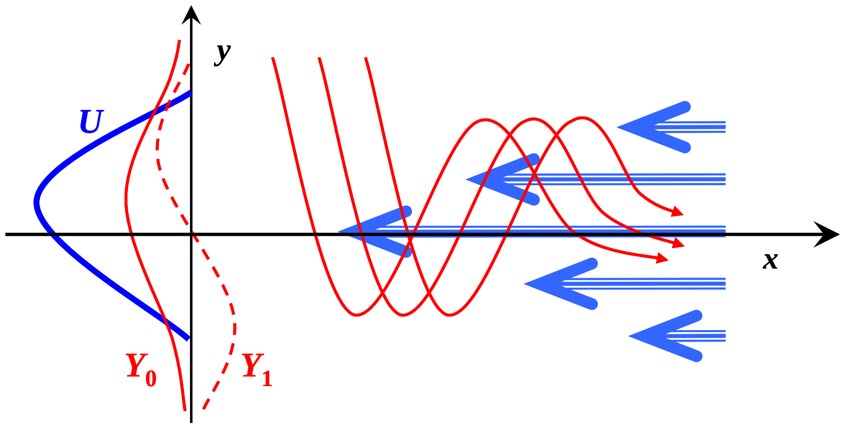
Surface elevation

$$\eta(x,y,t)=\sum_n A_n(x,t)Y_n(y)\cos(\omega_n t-kx)$$

Euler velocities

$$\vec{v}=\sum_n \vec{B}_n(x,t)\Phi_n(y,z)\cos(\omega_n t-kx)$$

Basic Equations



Incompressible ideal fluid

Euler equations

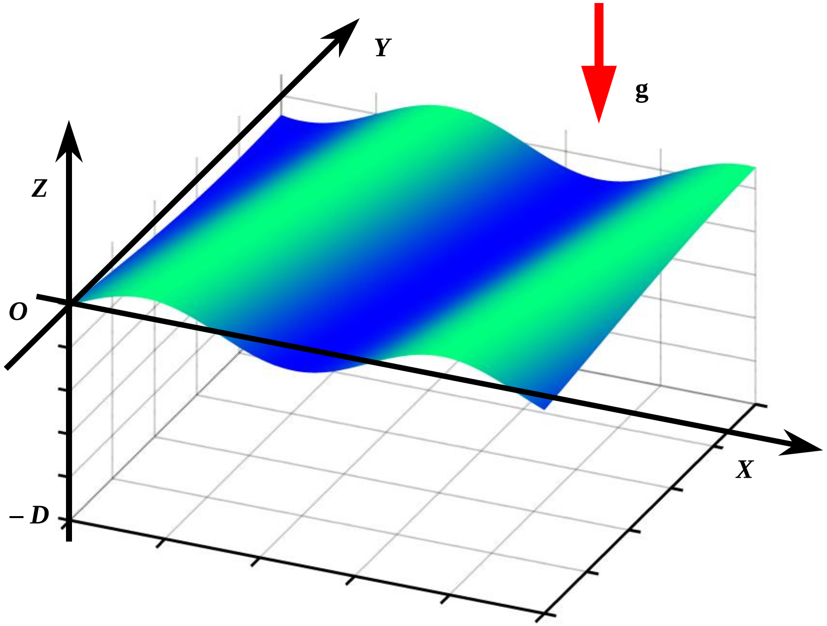
$$\frac{\partial \vec{v}}{\partial t} + \left(\vec{U} + \vec{v}, \nabla\right)\left(\vec{U} + \vec{v}\right) + \nabla P = \vec{g}$$

$$\nabla \cdot \left(\vec{U} + \vec{v}\right) = 0$$

$$P = 0 \quad at \quad z = \eta$$

$$\frac{\partial \eta}{\partial t} + \left(\vec{U} + \vec{v}, \nabla\right)\eta = v_z \quad at \quad z = \eta$$

$$v_z = 0 \quad at \quad z = -H$$



$$\boxed{\vec{U} = \left(U\left(y\right), 0, 0\right)}$$

Outline

Motivation

- Rogue wave on currents
- Wave on currents' theories
- The main idea

Problem statement

Linear theory

- Boundary-value problem
- Dispersion relation

Nonlinear theory

- 3-and 4-wave interactions

Discussion

- Wave enhancing effects

Outline

Motivation

- Rogue wave on currents
- Wave on currents theories
- The main idea

Problem statement

Linear theory

- Boundary-value problem and dispersion relation

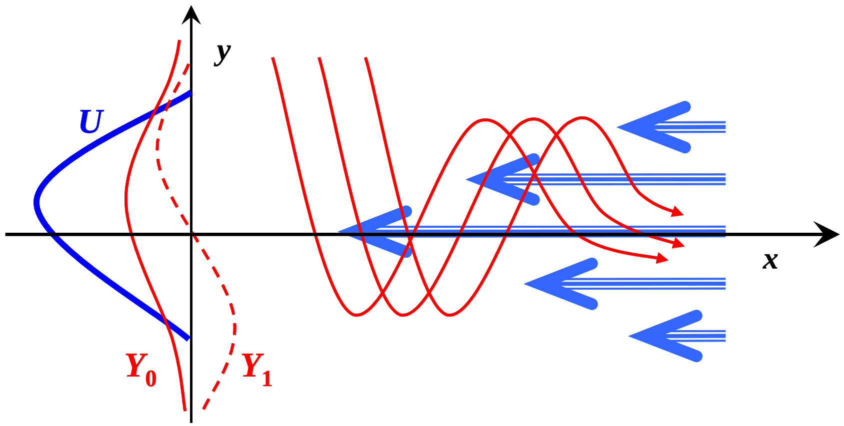
Nonlinear theory

- 3-wave interactions
- 4-wave interactions

Discussion

- Wave enhancing effects

Boundary-value problem



no extra assumptions

z - component of wave velocity, $w(y, z)$, has to satisfy

$$\frac{\partial^2 w}{\partial z^2} + \frac{\partial^2 w}{\partial y^2} + \left(\frac{\Omega''}{\Omega} - 2\frac{\Omega'^2}{\Omega^2} - k^2 \right) w = 0$$

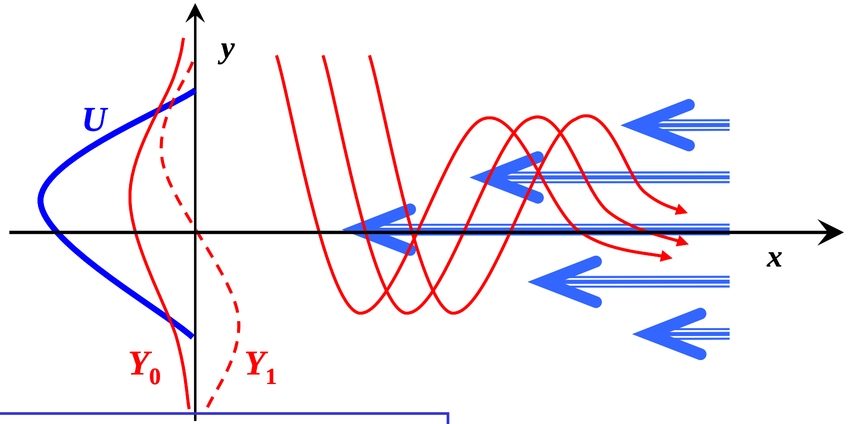
$$\Omega(y) = \omega - kU(y)$$

(analogue of the intrinsic frequency)

$$g\frac{\partial w}{\partial z} = \Omega^2 w, \qquad z = \eta$$

PDE boundary-value problem

Boundary-value problem



Seek solution in the form

$$w(y,z) = Y(y)Z(z,y)$$

$$\frac{\partial^2 Z}{\partial z^2} = h(y)^2 Z(z,y) \qquad Z(z \to -\infty) = 0 \qquad w \propto \exp[hz]$$

$$\frac{\partial^2 Y}{\partial y^2} + \left(\left\{ \frac{\Omega''}{\Omega} - 2\frac{\Omega'^2}{\Omega^2} \right\} + h^2 - k^2 \right) Y = -\left\{ 2\frac{1}{Z}\frac{\partial Z}{\partial y}\frac{\partial Y}{\partial y} + \frac{1}{Z}\frac{\partial^2 Z}{\partial y^2} Y \right\}$$

$$\Omega(y) = \omega - kU(y)$$

$$gh(y) = \Omega^2(y)$$

Asympotic solution of the BVP

1. Slow lateral variation of the current, $U = U(\mu y)$, $\mu << 1$
the longitudinal wavenumber $k \sim 1$

<u>The boundary-value problem</u>

$$w(y,z) = Y(y) Z(z, \mu y)$$

$$\frac{\partial^2 Z}{\partial z^2} = h(\mu y)^2 Z(z, \mu y)$$

h is a slow function of y

$$\frac{\partial^2 Y}{\partial y^2} + \left(\mu^2 \left\{\frac{\Omega''}{\Omega} - 2\frac{\Omega'^2}{\Omega^2}\right\} + h^2 - k^2\right) Y + \left\{2\mu\chi \frac{1}{Z}\frac{\partial Z}{\partial y}\frac{\partial Y}{\partial y} + \mu^2 \frac{1}{Z}\frac{\partial^2 Z}{\partial y^2} Y\right\} = 0$$

$$\Omega(\mu y) = \omega - kU(\mu y)$$

$$gh(\mu y) = \Omega^2(\mu y)$$

$$\frac{\partial Y}{\partial y} \propto \chi$$

Asymptotic solution of the BVP

1. Weak lateral variation of the current, $U = U(\mu y),\ \mu << 1$
the longitudinal wavenumber $k \sim 1$

The boundary-value problem

$$w(y,z) = Y(y)Z(z,\mu y) \qquad Y \xrightarrow[y\to\pm\infty]{} 0$$

$$\frac{\partial^2 Z}{\partial z^2} = h(\mu y)^2 Z(z,\mu y)$$

ω is the eigenvalue

$$\frac{\partial^2 Y}{\partial y^2} + \left(\frac{(\omega - kU(\mu y))^4}{g^2} - k^2\right)Y = O(\chi\mu, \mu^2), \qquad Y \xrightarrow[y\to\pm\infty]{} 0$$

Only one mode function $Y_n(y)$ corresponds to each eigenvalue ω_n

Modes $Y_n(y)$ are not necessarily orthogonal.

Asymptotic solutions

1. Slow lateral variation of the current, $U = U(\mu y),\ \mu << 1$
the longitudinal wavenumber $k \sim 1$

2. The current is moderately weak, $|kU|/\omega \sim \gamma, \quad \gamma << 1$

$$\frac{\partial^2 Y}{\partial y^2} + \left(\boxed{\frac{\omega^4}{g^2} - k^2} - 4\gamma \frac{k\omega^3 U}{g^2} \right) Y = O\left(\gamma^2, \chi\mu, \mu^2\right)$$

$$\left| 1 - \frac{\omega^4}{k^2 g^2} \right| >> O(\gamma)$$ travelling (passing through) waves

$$\frac{l^2}{k^2} \equiv 1 - \frac{\omega^4}{k^2 g^2} \le O(\gamma)$$ trapped modes

eigenvalues $l^2 > 0$ for trapped modes

Asymptotics of moderately weak current

1. Slow lateral variation of the current, $U = U(\mu y)$, $\mu << 1$
the longitudinal wavenumber $k \sim 1$

2. The current is moderately weak, $|kU| / \omega \sim \gamma$, $\gamma << 1$

Eigenvalue problem

$$\frac{\partial^2 Y}{\partial y^2} - \left(l^2 + 4\frac{k^3 U}{\sqrt{kg}} \right) Y = 0$$

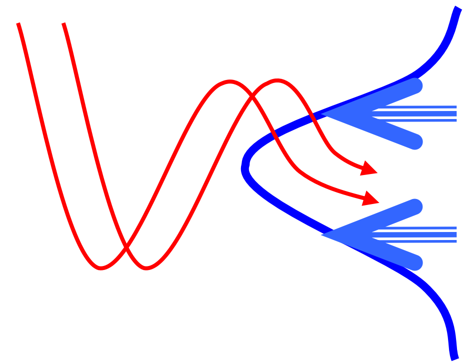
The classical stationary Schrodinger eq-n.

• Trapped modes **exist** when $kU < 0$ (the opposing current)
(There is always at least one trapped mode)

• **The eigenfunctions are orthogonal. They form a complete basis when passing through waves are taken into account.**

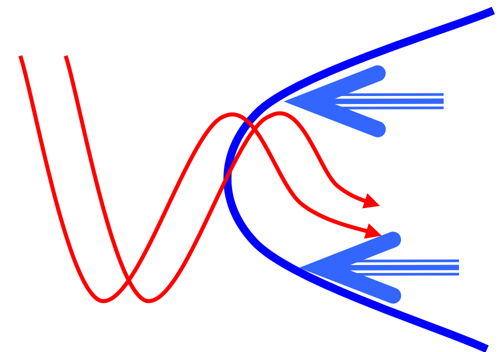
Model profiles [a set of exact solutions]

1. Sech^2 profile $\quad U = \dfrac{U_0}{\cosh^2(y/L)}$



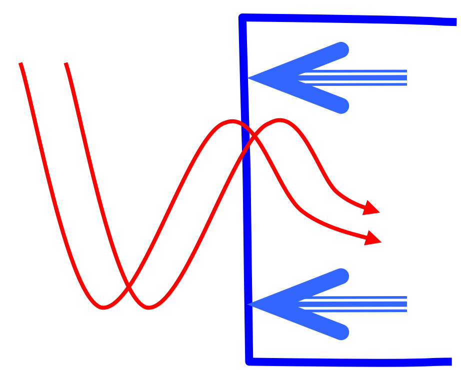
$$l^2 + 4\frac{k^3 U_0}{\sqrt{kg}} = -(2n+1)\frac{2k\sqrt{-kU_0}}{L\sqrt[4]{kg}} + \frac{(2n+1)^2}{4L^2}, \quad n = 0,1,..$$

2. Parabolic current $\quad U = \dfrac{U_0''}{2} y^2 + U_0$



$$l^2 + 4\frac{k^3 U_0}{\sqrt{kg}} = -(2n+1)\frac{k\sqrt{2kU_0''}}{\sqrt[4]{kg}}, \quad n = 0,1,..$$

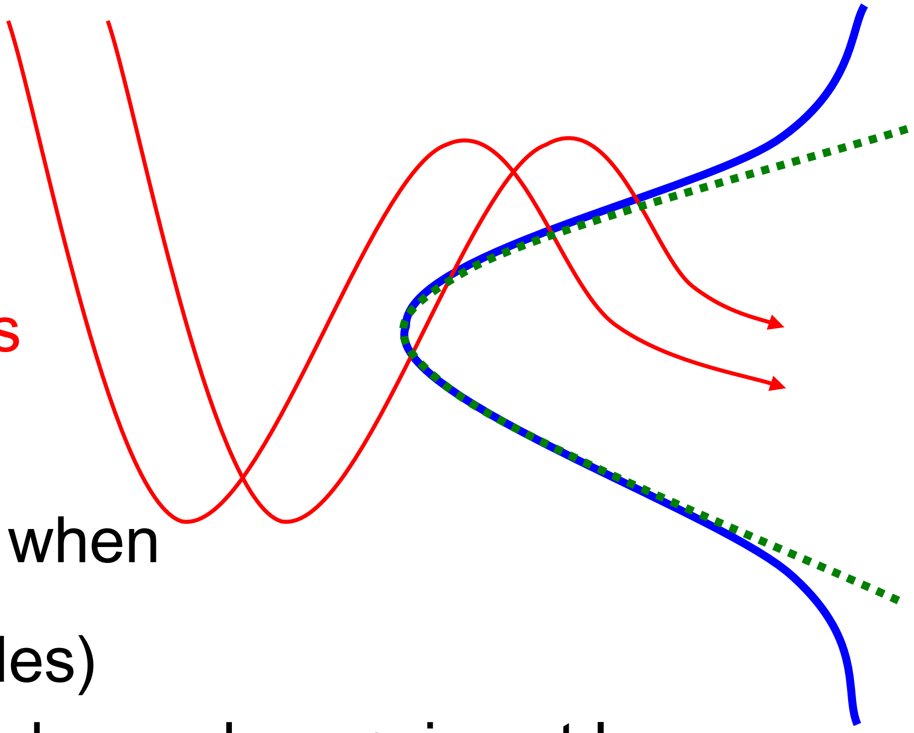
3. Top-hat current, width $2L$



$$l^2 \approx k^2 \left(\frac{4kU}{\sqrt{kg}} kL \right)^2$$ narrow current $\quad (kL)^2 \dfrac{|kU|}{\sqrt{kg}} << 1$

$$l^2 + 4\frac{k^3 U_0}{\sqrt{kg}} \approx \frac{\pi^2 (n+1)^2}{L^2}, \quad n = 0,1,..$$ otherwise

Asymptotics for the tip of the current

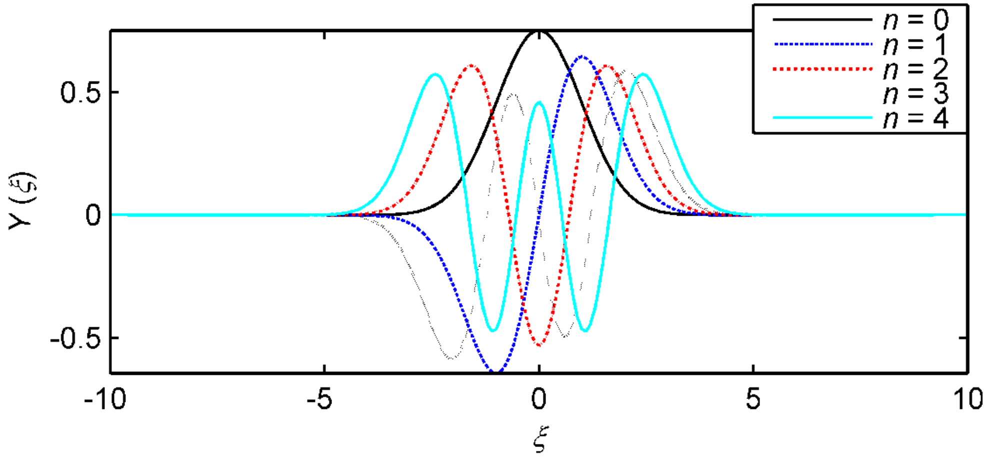


Parabolic shape
is a good approximation
for the tip of generic jet currents

Solutions for the parabolic
and the sech currents coincide when

$$(kL)^2 \frac{|kU|}{\sqrt{kg}} >> 1$$

(many modes)
and the mode number, *n*, is not large



Number of trapped modes

$$\frac{\partial^2 Y}{\partial y^2} - \left(l^2 + 4\frac{k^3 U}{\sqrt{kg}} \right) Y = 0$$

Bohr-Sommerfeld quantization rule for the quasi-classic limit

<u>Estimate of the number of trapped modes</u>

$$N_{tr} = \frac{2k}{\pi \sqrt[4]{kg}} \int_{-\infty}^{\infty} \underbrace{\sqrt{-kU}\,dy}_{} - \frac{1}{2} \approx 4 \int_{-\infty}^{\infty} \sqrt{-\frac{U}{C_{ph}}} \frac{dy}{\lambda}$$

$$C_{ph} = \sqrt{g/k}$$

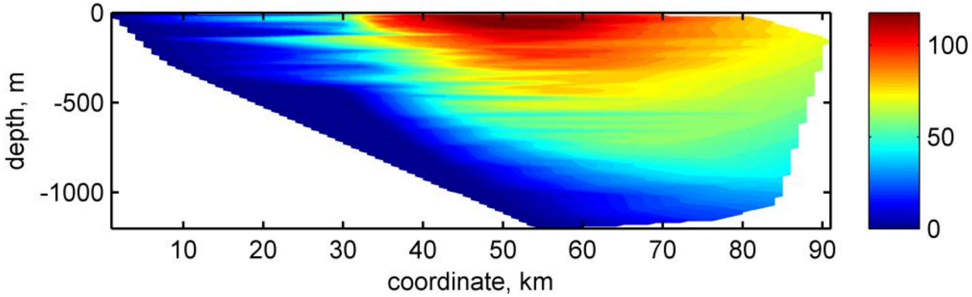
$$\lambda = \frac{2\pi}{k}$$

'Field study': Agulhas current

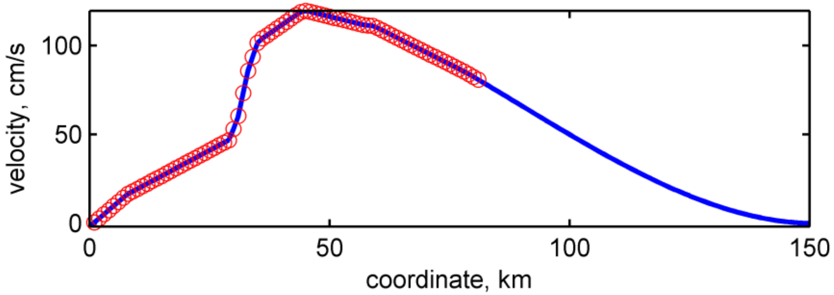
$$N_{tr} \approx 4 \int_{-\infty}^{\infty} \sqrt{-\frac{U}{C_{ph}}} \frac{dy}{\lambda}$$

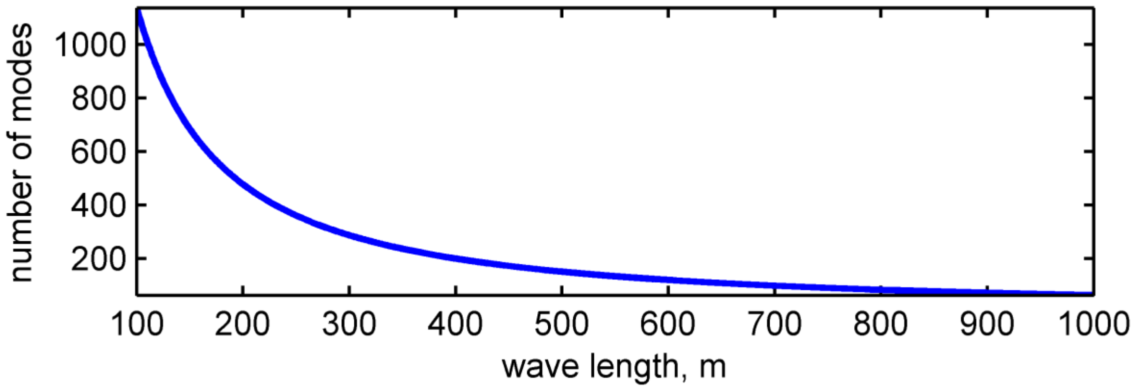
Current profile

velocity



current profile

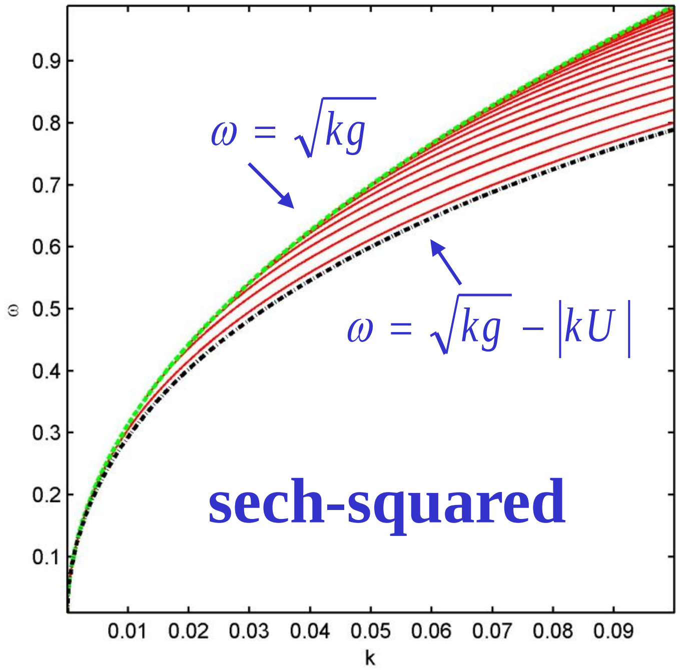




$O(10^2)$ trapped modes

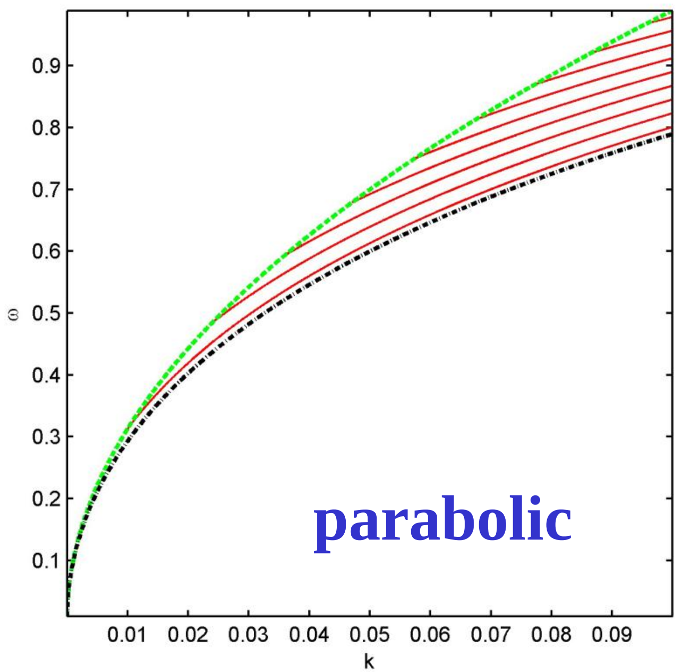
Dispersion relation

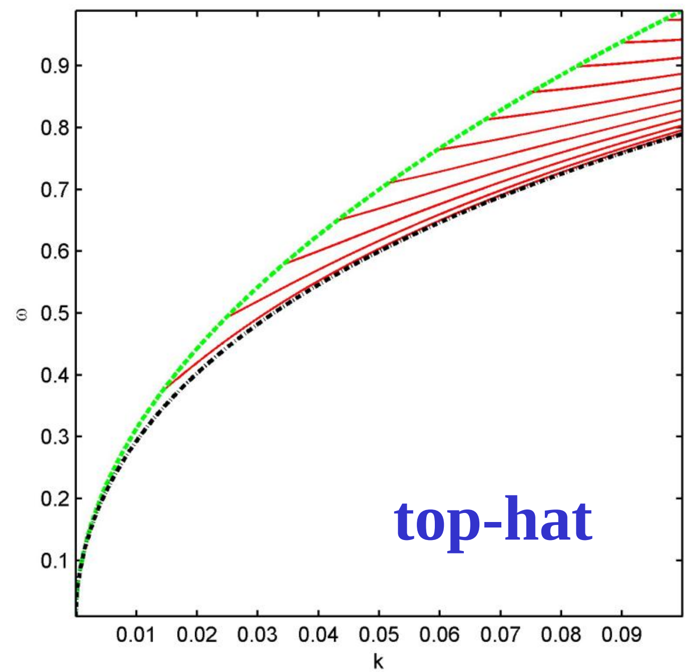
(parameters typical of rip currents)



$$\omega_n = \sqrt{kg}\left(1 - \frac{1}{4}\frac{l^2}{k^2}\right), \quad l^2 > 0$$

l^2 is the solution of the BVP





Outline

Motivation

- Rogue wave on currents
- Wave on currents' theories
- The main idea

Problem statement

Linear theory

- Boundary-value problem + dispersion relation

Nonlinear theory

- 3- and 4-wave interactions

Discussion

- Wave enhancing effects

Outline

Motivation

- Rogue wave on currents
- Wave on currents theories
- The main idea

Problem statement

Linear theory

- Boundary-value problem
- Dispersion relation

Nonlinear theory

- 3- and 4-wave interactions

Discussion

- Wave enhancing effects

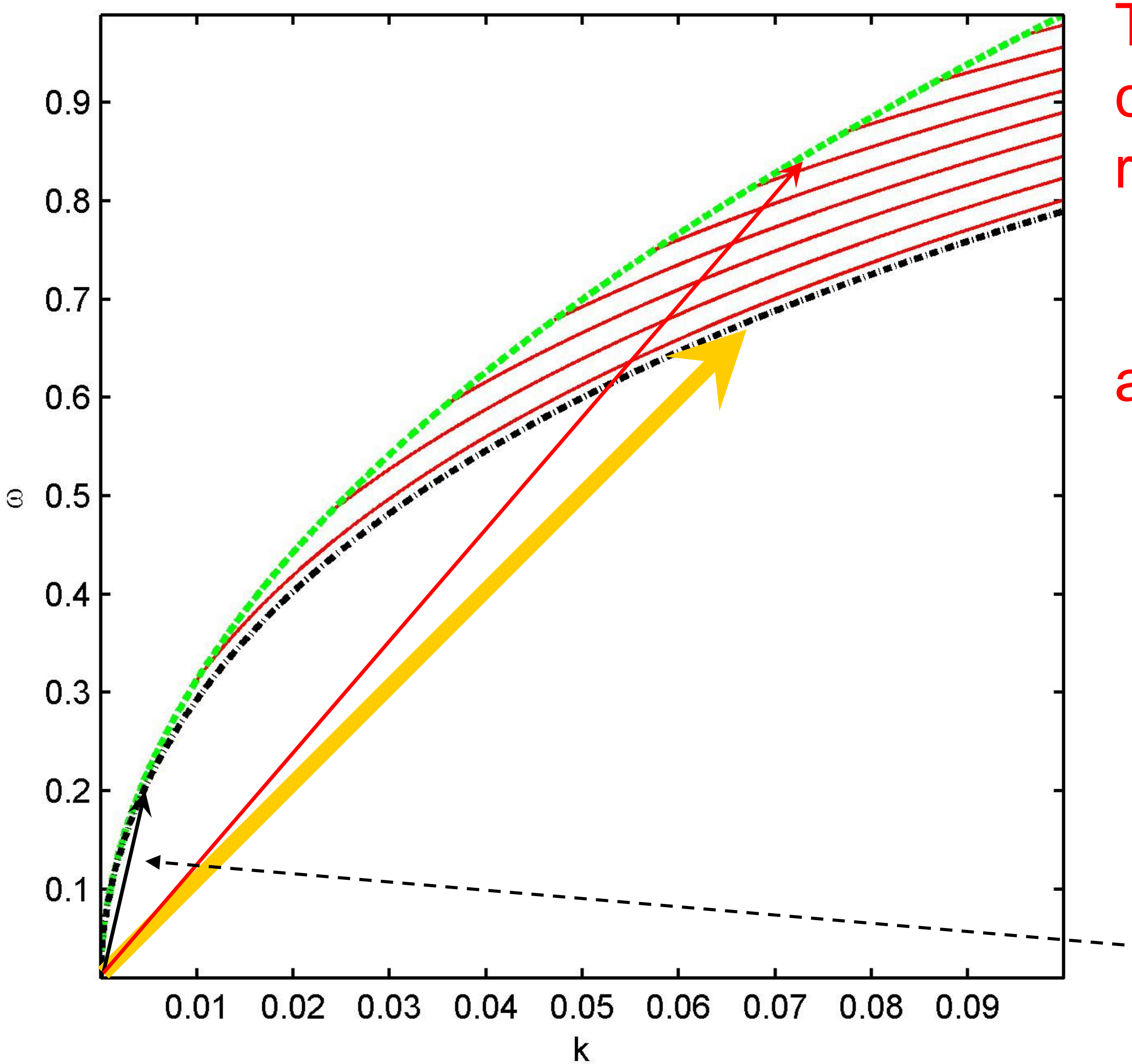
Nonlinear dynamics

The trapped modes differ qualitatively from the free waves. This fact profoundly changes all aspects of their nonlinear dynamics.

Here we focus upon wave resonant interactions

Triad wave resonances

3-wave resonances for trapped waves are always allowed



The resonance conditions for 3-wave resonances

$$k_2 - k_1 = k_3$$

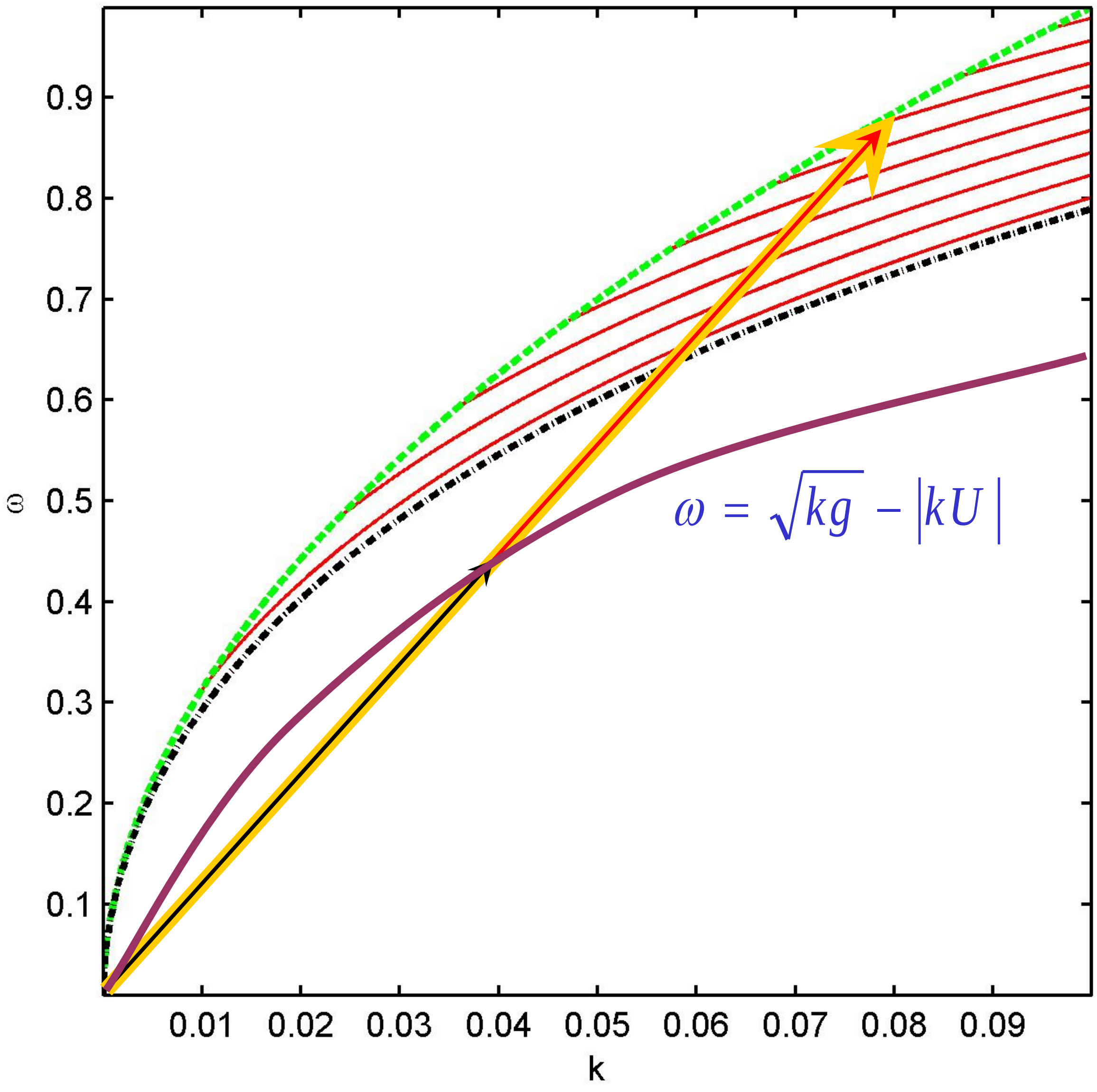
$$\omega_2 - \omega_1 = \omega_3$$

are satisfied

$$4\sqrt{\frac{k_3}{k_1}} = \frac{l_1^2}{k_1^2} - \frac{l_2^2}{k_2^2} \propto \gamma$$

The third wave must be long

Triad wave resonances



3-wave resonances

$$\omega + \omega = 2\omega$$

$$k + k = 2k$$

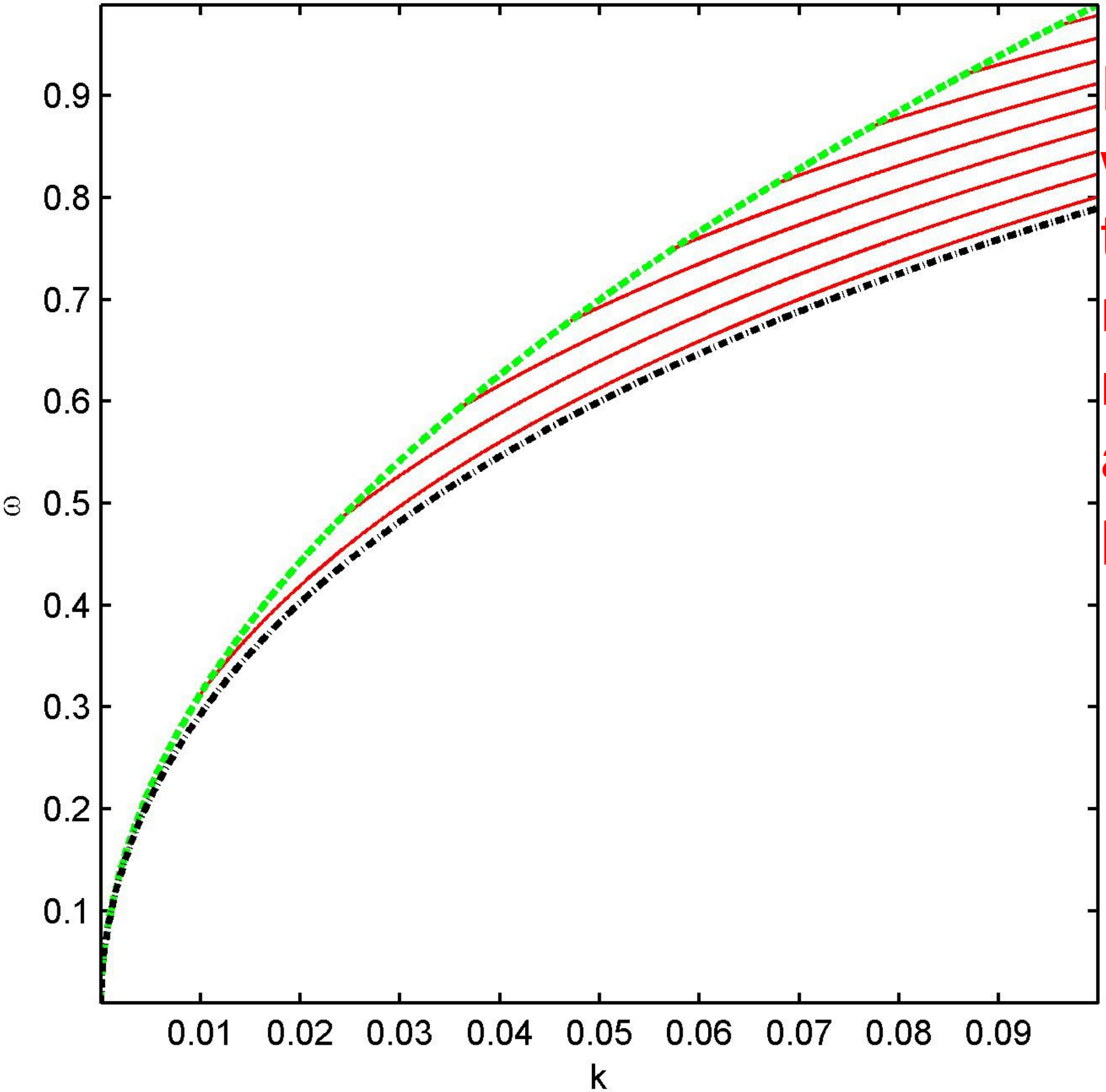
are possible for waves of comparable scales if

$$\left|\frac{kU}{\omega}\right| > 1 - \frac{1}{\sqrt{2}} \approx 0.3$$

Current should be strong enough: $\left|\frac{kU}{\omega}\right| \sim 1$

Wave resonances

4-wave resonances between trapped waves of comparable scales are always allowed for currents of arbitrary strength



In contrast to free wave waves non-trivial (and non-degenerate) resonances are allowed for 1d propagation

$$k_1 + k_2 = k_3 + k_4$$

$$\omega_1 + \omega_2 = \omega_3 + \omega_4$$

New type of interactions: Triad interactions between trapped and passing through waves

If we consider a pair of trapped modes $k_1, k_2,$ and a passing through wave $k_3,$ then the triad resonance conditions, e.g.,

$$k_1 + k_2 = k_3, \qquad \omega_1 + \omega_2 = \omega_3,$$

are easy to satisfy. From the perspective of trapped modes, the passing through waves have infinite energy. This very special interaction leads to unlimited growth of trapped waves.

Nonlinear equations

$$\frac{\partial \vec{v}}{\partial t} + (\vec{U} + \vec{v}, \nabla)(\vec{U} + \vec{v}) + \nabla P = \vec{g} \qquad \vec{U} = (U(y), 0, 0)$$

$$\nabla \cdot (\vec{U} + \vec{v}) = 0$$

$$P = 0 \quad at \quad z = \eta$$

$$\frac{\partial \eta}{\partial t} + (\vec{U} + \vec{v}, \nabla)\eta = v_z \quad at \quad z = \eta$$

$$v_z = 0 \quad at \quad z = -H$$

Weakly nonlinear asymptotic theory

$$k\eta \sim \varepsilon, \quad (u, v, w) \sim \varepsilon C_{ph} \quad \varepsilon << 1$$

current is slow function of *y* (μ << 1)

Asymptotic Series

Weakly nonlinear asymptotic theory

$$k\eta \sim \varepsilon, \quad (u, v, w) \sim \varepsilon C_{ph}, \quad \varepsilon << 1$$

$$\eta(x, y, t) = \varepsilon \eta^{[1]} + \varepsilon^2 \eta^{[2]} + \varepsilon^3 \eta^{[3]} + \ldots$$

$$u(x, y, z, t) = \varepsilon u^{[1]} + \varepsilon^2 u^{[2]} + \varepsilon^3 u^{[3]} + \ldots$$

$$v(x, y, z, t) = \varepsilon v^{[1]} + \varepsilon^2 v^{[2]} + \varepsilon^3 v^{[3]} + \ldots$$

$$w(x, y, z, t) = \varepsilon w^{[1]} + \varepsilon^2 w^{[2]} + \varepsilon^3 w^{[3]} + \ldots$$

Slow time $T = \sum_{m=1}^{\infty} \varepsilon^m t$ Slow coordinate $\frac{1}{k}\frac{\partial}{\partial X} \propto \varepsilon$

Ansatz

Weakly nonlinear asymptotic theory

$$k\eta \sim \varepsilon, \qquad (u, v, w) \sim \varepsilon C_{ph}, \qquad \varepsilon << 1$$

$$w^{[1]} = \frac{1}{2}\sum_r \left(B_r\left(X, T\right)\exp\left(i\omega_r t - ik_r t\right) + c.c.\right)Y_r\left(y\right)Z_r\left(z\right)$$

$$Z_r\left(z\right) = \exp\left(h_r\left(y\right)z\right)$$

Weakly nonlinear asymptotic theory

$$k\eta \sim \varepsilon, \quad (u, v, w) \sim \varepsilon C_{ph}, \quad \varepsilon << 1$$

Solution in the order ε^1

$$w^{[1]} = \frac{1}{2}\sum_r \left(B_r(X,T)\exp(i\omega_r t - ik_r t) + c.c.\right)Y_r(y)Z_r(z)$$

$$u^{[1]} = \frac{1}{2}\sum_r \left(-\frac{k_r}{h_r}B_r \exp(i\omega_r t - ik_r t) + c.c.\right)Y_r Z_r$$

$$v^{[1]} = \frac{1}{2}\sum_r \left(\frac{1}{h_r}B_r \exp(i\omega_r t - ik_r t) + c.c.\right)\frac{\partial Y_r}{\partial y}Z_r$$

$$\eta^{[1]} = \frac{1}{2}\sum_r \left(A_r \exp(i\omega_r t - ik_r t) + c.c.\right)Y_r(y) \qquad A_r = -i\frac{\Omega_r}{gh_r}B_r$$

$$h_r = \frac{\Omega_r^2}{g}$$ are subjects for the BVP obtained earlier

$$\Omega = \omega - kU$$

Weakly nonlinear asymptotic theory

$$k\eta \sim \varepsilon, \qquad (u, v, w) \sim \varepsilon C_{ph}, \qquad \varepsilon << 1$$

Solution in the order ε^2

$$w^{[2]} = \frac{1}{2}\sum_r \left(i \frac{k_r}{h_2} \frac{\partial B_r}{\partial X} \exp\left(i\omega_r t - ik_r t\right) + c.c. \right) Y_r z Z_r +$$

$$+ \frac{1}{2}\sum_f \sum_r \left(B_r^f \exp\left(i\omega_r t - ik_r t\right) + c.c. \right) Y_r^f Z_r^f$$

forced components

$$h_r^f = \frac{\left(\omega^f - k_r U\right)^2}{g} \qquad \omega^f \neq \omega_r$$

$$h_r = \frac{\left(\omega - k_r U\right)^2}{g}$$

are subjects for the BVP obtained earlier

Equations for 3-wave interactions

$$\frac{\partial B_1}{\partial t}Y_1 + V_1\frac{\partial B_1}{\partial X}Y_1 + \sum_f \kappa_1^f B_1^f Y_1^f + \sigma_1 B_2^* B_3 \frac{\partial Y_2}{\partial y}\frac{\partial Y_3}{\partial y} + \rho_1 B_2^* B_3 Y_2 Y_3 = 0$$

$$\frac{\partial B_2}{\partial t}Y_2 + V_2\frac{\partial B_2}{\partial X}Y_2 + \sum_f \kappa_2^f B_2^f Y_2^f + \sigma_2 B_1 B_3 \frac{\partial Y_1}{\partial y}\frac{\partial Y_3}{\partial y} + \rho_2 B_1 B_3 Y_1 Y_3 = 0$$

$$\frac{\partial B_3}{\partial t}Y_3 + V_3\frac{\partial B_3}{\partial X}Y_3 + \sum_f \kappa_3^f B_3^f Y_3^f + \sigma_3 B_1^* B_2 \frac{\partial Y_1}{\partial y}\frac{\partial Y_2}{\partial y} + \rho_3 B_1^* B_2 Y_1 Y_2 = 0$$

Integration in the lateral dimension. Orthogonality of eigenmodes

$$\boxed{\frac{\partial B_1}{\partial t} + V_1\frac{\partial B_1}{\partial X} + \nu_1 B_2^* B_3 = 0}$$

$$\nu_1 = \frac{\int_{-\infty}^{\infty}\sigma_1 Y_1 Y_2' Y_3' dy}{\int_{-\infty}^{\infty} Y_1^2 dy} + \frac{\int_{-\infty}^{\infty}\rho_1 Y_1 Y_2 Y_3 dy}{\int_{-\infty}^{\infty} Y_1^2 dy}$$

$$V_1 = \int_{-\infty}^{\infty}\left(\frac{k_1 g^2}{2(\omega_1 - k_1 U)^3} + U\right) Y_1^2 dy$$

The approach is efficient if modes are [weakly] non-orthogonal

Equations for 3-wave interactions

$$\frac{\partial B_1}{\partial t} + V_1 \frac{\partial B_1}{\partial X} + \nu_1 B_2^* B_3 = 0$$

$$\frac{\partial B_2}{\partial t} + V_2 \frac{\partial B_2}{\partial X} + \nu_2 B_1 B_3 = 0$$

$$\frac{\partial B_3}{\partial t} + V_3 \frac{\partial B_3}{\partial X} + \nu_3 B_1^* B_2 = 0$$

- No assumption on the current magnitude was made

- Linear wave group velocity

$$V_r = \int_{-\infty}^{\infty} \left(\frac{k_r g^2}{2(\omega_r - k_r U)^3} + U \right) Y_r^2 dy$$

It tends to

$$V_r = \frac{\Omega_r}{2k_r} + U$$

if waves propagate almost along the current

Equations for 3-wave interactions

$$\frac{\partial B_1}{\partial t} + V_1 \frac{\partial B_1}{\partial X} + \nu_1 B_2^* B_3 = 0$$
$$\frac{\partial B_2}{\partial t} + V_2 \frac{\partial B_2}{\partial X} + \nu_2 B_1 B_3 = 0$$
$$\frac{\partial B_3}{\partial t} + V_3 \frac{\partial B_3}{\partial X} + \nu_3 B_1^* B_2 = 0$$

3-wave interactions are essentially non-potential. Interaction coefficients depend on vorticity of the current

Assumption of a moderately weak current, $\gamma << 1$

Normal form of the system

$$\frac{\partial b_1}{\partial t} + V_1 \frac{\partial b_1}{\partial X} + \nu b_2^* b_3 = 0$$
$$\frac{\partial b_2}{\partial t} + V_2 \frac{\partial b_2}{\partial X} + \nu b_1 b_3 = 0$$
$$\frac{\partial b_3}{\partial t} + V_3 \frac{\partial b_3}{\partial X} + \nu b_1^* b_2 = 0$$

$$\nu = \sqrt{\nu_1 \nu_3 \nu_3} = O\left(\gamma^{3/2}\right) \qquad B_r = b_r \sqrt{\nu_r}$$

weak interaction for weak currents

Equations for 4-wave interactions

$$k_1 + k_2 = k_3 + k_4 \qquad k_1 \geq k_3 \geq k_4 \geq k_2$$

$$\omega_1 + \omega_2 = \omega_3 + \omega_4$$

Under assumption of smooth ($\mu << 1$) and moderately weak current ($\gamma << 1$) only free (master) modes should be considered

Equations for 4-wave interactions

Wave quartet (r = 1, 2, 3, 4)

$$i\sqrt{gk_1}\frac{\partial B_1}{\partial t} = \alpha_1 B_1 |B_1|^2 + \sum_{j=2,3,4} \alpha_{1j} B_1 |B_j|^2 + \beta_1 B_2^* B_3 B_4$$

$$i\sqrt{gk_2}\frac{\partial B_2}{\partial t} = \alpha_2 B_2 |B_2|^2 + \sum_{j=1,3,4} \alpha_{2j} B_2 |B_j|^2 + \beta_2 B_1^* B_3 B_4$$

$$i\sqrt{gk_3}\frac{\partial B_3}{\partial t} = \alpha_3 B_3 |B_3|^2 + \sum_{j=1,2,4} \alpha_{1j} B_3 |B_j|^2 + \beta_3 B_1 B_2 B_4^*$$

$$i\sqrt{gk_4}\frac{\partial B_4}{\partial t} = \alpha_4 B_4 |B_4|^2 + \sum_{j=1,2,3} \alpha_{4j} B_4 |B_j|^2 + \beta_1 B_1 B_2 B_3^*$$

Nonlinear coefficients

$$i\sqrt{gk_1}\frac{\partial B_1}{\partial t} = \alpha_1 B_1 |B_1|^2 + \sum_{j=2,3,4} \alpha_{1j} B_1 |B_j|^2 + \beta_1 B_2^* B_3 B_4$$

Nonlinear coefficients

$$\alpha_r = \frac{k_r^2}{2} I_r \qquad \alpha_{rq} = 2\alpha_r I_{n_r n_q} \begin{cases} \sqrt{k_r / k_q}, & k_q > k_r \\ \sqrt{k_q / k_r}, & k_q < k_r \end{cases} \qquad \alpha_r \le \alpha_{rq}$$

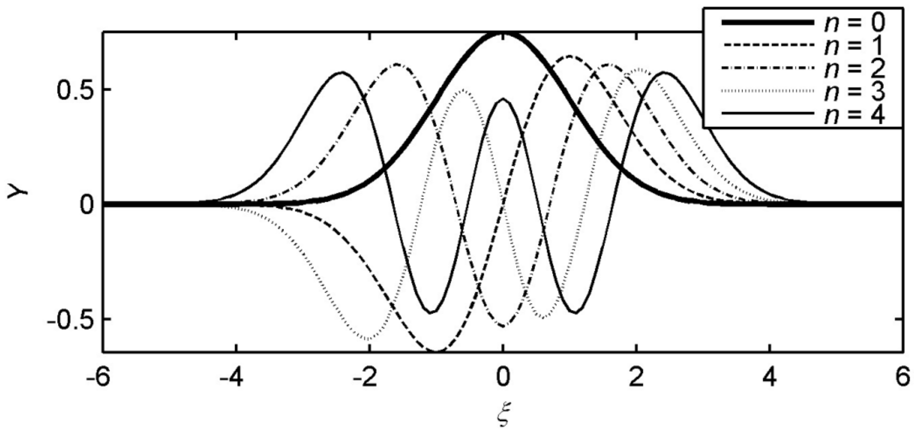
$$\beta_r = \Gamma(k_1, k_2, k_3, k_4) \qquad \lim_{k_3 \to k_1, k_4 \to k_2} \beta_r = 2\alpha_r J_r \sqrt{k_2 / k_1}$$

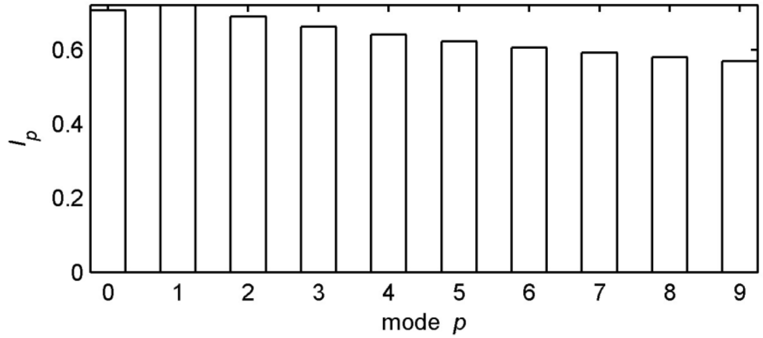
Mode overlap integrals

$$I_n = \frac{\int_{-\infty}^{\infty} Y_n^4 dy}{\int_{-\infty}^{\infty} Y_n^2 dy} \qquad I_{nm} = \frac{\int_{-\infty}^{\infty} Y_n^2 Y_m^2 dy}{\int_{-\infty}^{\infty} Y_n^2 dy} \qquad J_r = \frac{\int_{-\infty}^{\infty} Y_{n_1} Y_{n_2} Y_{n_3} Y_{n_4} dy}{\int_{-\infty}^{\infty} Y_{n_r}^2 dy}$$

Mode overlap integrals

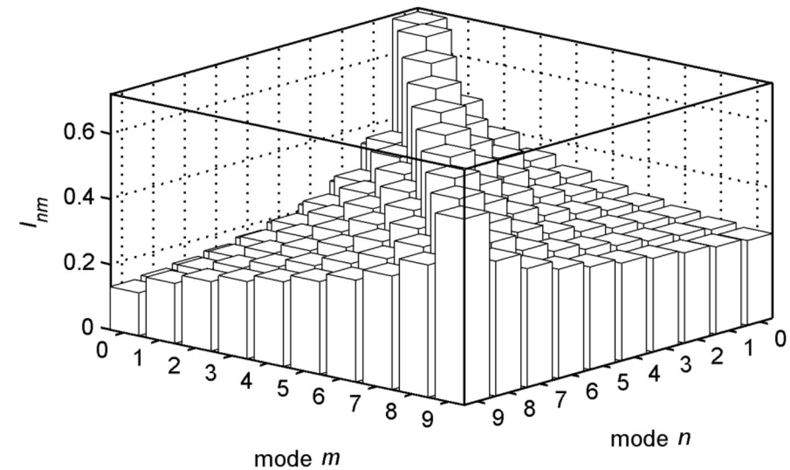
Parabolic current



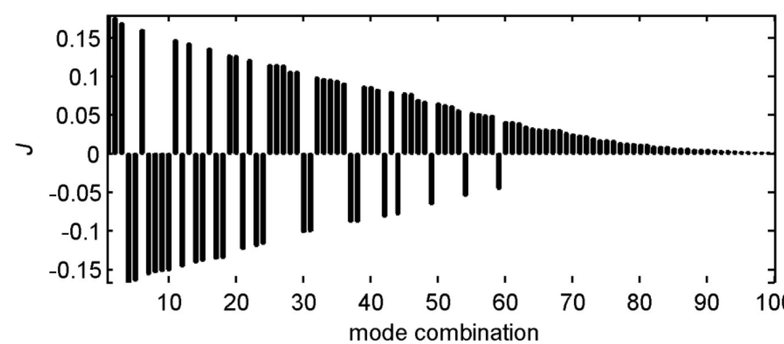


one mode self-interaction

two-mode interaction



3- and 4- different mode interaction



Equations

$$i\sqrt{gk_1}\,\frac{\partial B_1}{\partial t} = \alpha_1 B_1 |B_1|^2 + \sum_{j=2,3,4} \alpha_{1j} B_1 |B_j|^2 + \beta_1 B_2^* B_3 B_4$$

$$i\sqrt{gk_2}\,\frac{\partial B_2}{\partial t} = \alpha_2 B_2 |B_2|^2 + \sum_{j=1,3,4} \alpha_{2j} B_2 |B_j|^2 + \beta_2 B_1^* B_3 B_4$$

$$i\sqrt{gk_3}\,\frac{\partial B_3}{\partial t} = \alpha_3 B_3 |B_3|^2 + \sum_{j=1,2,4} \alpha_{1j} B_3 |B_j|^2 + \beta_3 B_1 B_2 B_4^*$$

$$i\sqrt{gk_4}\,\frac{\partial B_4}{\partial t} = \alpha_4 B_4 |B_4|^2 + \sum_{j=1,2,3} \alpha_{4j} B_4 |B_j|^2 + \beta_1 B_1 B_2 B_3^*$$

Nonlinear dynamics of trapped waves is controlled by both triad and quartet interactions.

Depending on the parameters of the current and width of the wave spectrum either triad or quartet might be dominant, or both could be equally important. There is a variety of scenarios to be explored.

Interaction with the passing through waves results in constant pumping of energy into trapped modes.

For the narrowband wave spectra we arrive at the classical 1-d NLS situation **which is now robust**.

Outline

Motivation

- Freak wave on currents
- Wave on currents' theories
- The main idea

Problem statement

Linear theory

- Boundary-value problem
- Dispersion relation

Nonlinear theory

- 3-wave interactions
- 4-wave interactions

Discussion

- Wave enhancing effects

Outline

Motivation

- Rogue wave on currents
- Wave on currents’ theories
- The main idea

Problem statement

Linear theory

- Boundary-value problem
- Dispersion relation

Nonlinear theory

- 3-wave interactions
- 4-wave interactions

Discussion

- Wave enhancing effects

Conclusions

Description of waves on jet currents:

●In linear setting there are always trapped modes. BVP is solved asymptotically. The trapped waves differ qualitatively from freely propagating waves

For weakly nonlinear waves:

● 3-wave resonances always occur. Interactions are non-potential.

● nonlinear evolution equations represent coupled systems of equations (triads and quartets)

Particular cases: - coupled 1-d NLS equations,

or - 1-d NLS equation.

● Interaction with passing through waves leads to constant pumping of energy into the trapped modes.

Conclusions

Effectively 1D dynamics of 3D waves =>

- 1D propagation is imposed by the physical nature of trapped modes. No need in narrow angular spectrum assumptions. A true 1D NLS framework for water waves

- unmitigated effects of modulational instability

- There is a principal possibility of the rogue wave deterministic forecasting due to the integrability of the NLS model or proximity to an integrable system of more general 1d models

Generalizations:

Other types of waveguides:

- Topographic (e.g. bars) .
- Combined (topography + current) , e.g. longshore currents and nearshore environment.
- Other geometries (e.g. vortices rather than jets).
- Other types of waves (e.g. internal waves)

Rogue wave implications

Linear and nonlinear effects which increase probability of freak waves on jet currents

- **Effectively 1D wave dynamics + decrease of group velocity => better conditions for the modulational instability onset and development**

- **Adiabatic non-uniformity of the current**
  - **wave steepening due to change of group velocity**
  - **with extra mode localization (focusing) at stronger currents**

– additional amplification of nonlinear wave groups

- **– rise of the ceiling: increase of the maximal allowed amplitude (before breaking) due to 3-d structure of the wave**

References

V.Shrira, A.Slunyaev (2014) *Trapped waves on jet currents: asymptotic modal approach*, J. Fluid Mech., **738**, 65-104.

V.Shrira, A.Slunyaev (2014) Nonlinear dynamics of trapped waves on jet currents and rogue waves, Phys. Review E, **89**, 041002(R).

Thank you!